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Convergence of a gradient damage model toward a cohesive zone model

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ABSTRACT

The study starts from a specific gradient damage model which admits a closed-form solution in the case of uniaxial tension. It enables to separate the parameters of the model between a length scale, characteristic of nonlocal effects, and macroscopic parameters which retain their meaning in a cohesive crack setting. A convergence analysis is performed: the response of a cohesive zone model is retrieved when the length scale goes to zero while keeping the macroscopic parameters constant.

1. Introduction

Continuum Damage Mechanics (CDM) appears to be an attractive theory to model brittle fracture from damage inception up to complete structural failure [1]. Unfortunately, damage localisation induced by strain-softening [2] requires the introduction of some form of nonlocality into the constitutive relation which makes the theory by far more complex. On the other hand, Cohesive Zone Models (CZM) are well adapted to describe crack initiation and propagation, at least for given crack paths, since they rely on a discontinuous setting which summarises the damage phenomena [3]. Moreover, they are a natural extension of Linear Fracture Mechanics [4] and hence they are paid increasing attention in industrial applications. At first sight, the CDM and CZM approaches thus seem quite disconnected; the purpose of this work consists in somewhat reconciling them.

The study starts from a gradient damage model [5,6], which benefits from a closed-form solution in a one-dimensional context [7]. The internal parameters of the model can be related to physical ones: the elastic stiffness, the fracture energy, the peak stress, the critical opening displacement and the damage band width. The latter – a length scale – is characteristic from nonlocal damage while the former ones remain meaningful in a discontinuous setting (they are hence said "macroscopic"). This suggests analysing the convergence of the solution for vanishing length scale while keeping the macroscopic parameters constant. It is showed that the limit exists in 1D and that it corresponds to the response of a cohesive zone model, hence exhibiting a relation between both formulations.

The gradient damage model is presented in the next section and its closed-form solution is recalled. In Section 3, the solution is rewritten in order to separate the influence of the length scale from the effect of the macroscopic parameters. This finally enables to derive the asymptotic response for vanishing length scale.

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2. A brittle gradient damage model

2.1. Constitutive law

The analysis is led in the context of isotropic brittle damage. The material state is defined by the strain tensor $\boldsymbol{\varepsilon}$ and a scalar damage variable a. The stress–strain relation is brittle elastic, where damage progressively weakens the initial stiffness:

$$\boldsymbol{\sigma} = A(a)\mathbf{E} : \boldsymbol{\varepsilon} \tag{1}$$

The stress is denoted $\boldsymbol{\sigma}$, $\mathbf{E}$ is Hooke's tensor (corresponding to Young's modulus E and Poisson's ratio ν) and $0 \leqslant A(a) \leqslant 1$ is the stiffness function. Note that the stress–strain relation is linear, which implies that the model does not take into account possible crack closure. The results mentioned in the introduction are obtained with a specific stiffness function the expression of which is given right now for the sake of clarity:

$$A(a) = \frac{(1-a)^2}{1 + (m-2)a + (1+pm)a^2} \tag{2}$$

The motivation for such an expression is explained in Section 2.3 on the basis of physical and numerical arguments. In particular, the internal parameters m and p are restricted to $p \geqslant 1$ and $m \geqslant p + 2$. It can be noticed that $a = 0$ corresponds to the sound material and $a = 1$ corresponds to ultimate damage: $A(0) = 1$, $A(1) = 0$.

The damage driving force Y derives from the elastic energy; it reads:

$$Y = -\frac{1}{2} A'(a) \boldsymbol{\varepsilon} : \mathbf{E} : \boldsymbol{\varepsilon} \tag{3}$$

In order to control the localisation of strain and damage, a coupling between neighbour material points is modelled through the introduction of the Laplacian of damage $\nabla^2 a$ into the yield function:

$$f\left(Y, \nabla^2 a\right) = Y + c\nabla^2 a - k \tag{4}$$

where $k > 0$ is a yield threshold and $c > 0$ is a parameter which governs the intensity of the coupling, hence the strength of nonlocal effects. The Kuhn–Tucker consistency condition takes its usual form:

$$f \leqslant 0; \qquad \dot{a} \geqslant 0; \qquad \dot{a} f = 0 \tag{5}$$

where the dot denotes time differentiation. Finally, an additional boundary condition and interface conditions have to be postulated. They consist of a regularity condition of the damage field and natural boundary conditions. If $\partial\Omega$ denotes the boundary of the body domain Ω, $\mathbf{n}$ its outer normal, Γ a potential surface of discontinuity, $\boldsymbol{\nu}$ its normal and $[\![\cdot]\!]$ the discontinuity of a quantity across Γ, then the interface conditions and the boundary condition respectively read:

$$[\![a]\!] = 0 \quad \text{and} \quad [\![c\nabla a]\!] \cdot \boldsymbol{\nu} = 0 \quad \text{across } \Gamma \tag{6}$$

$$\nabla a \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \tag{7}$$

The model belongs to the class of generalised standard materials, extended to gradient constitutive law [5]. The corresponding Helmholtz' free energy $\mathcal{F}$ and the dissipation potential $\mathcal{D}$ respectively read:

$$\mathcal{F}(\boldsymbol{\varepsilon}, a) = \int_{\Omega} \frac{1}{2} A(a) \boldsymbol{\varepsilon} : \mathbf{E} : \boldsymbol{\varepsilon} + \frac{c}{2} \nabla a \cdot \nabla a; \qquad \mathcal{D}(\dot{a}) = \int_{\Omega} k\dot{a} + \mathcal{I}_{\mathbb{R}^+}(\dot{a}) \tag{8}$$

with $\mathcal{I}_{\mathbb{R}^+}$ the indicator function that enforces $\dot{a} \geqslant 0$ [6,7].

2.2. Closed-form solution

From now on, we focus our attention on a one-dimensional problem. A bar is submitted to tension until damage occurs. Damage is then assumed to localise, regardless of stability conditions [8]; in particular, the bar is sufficiently large so that its boundaries do not interfere with the localisation band. Let us recall the results obtained in [7].

The profile of the damage band is given by the following implicit expression:

$$x(a_0, a) = \int_a^{a_0} G(a_0, s)^{-1/2}\, \mathrm{d}s \quad \text{with } G(a_0, a) = \frac{2ka_0}{c} \left[\frac{a}{a_0} - \frac{A(a)^{-1} - 1}{A(a_0)^{-1} - 1} \right] \tag{9}$$

where a is the damage at the point of co-ordinate $x = x(a_0, a)$ and a_0 is the current top damage value inside the band. The latter is used to parameterise the response of the bar, with $0 \leqslant a_0 \leqslant 1$. Without loss of generality, the damage band is

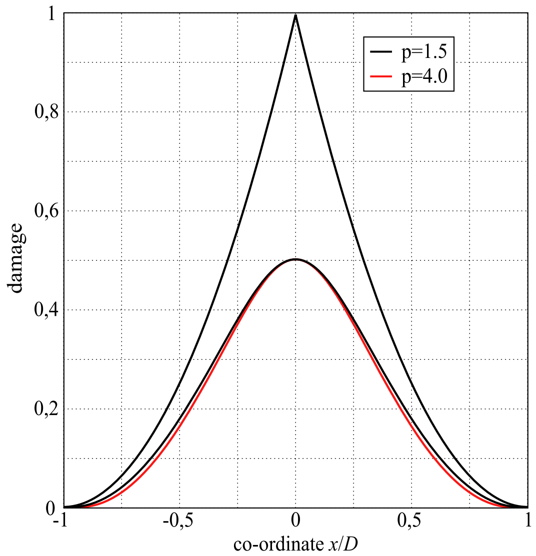

Fig. 1. Damage profile.

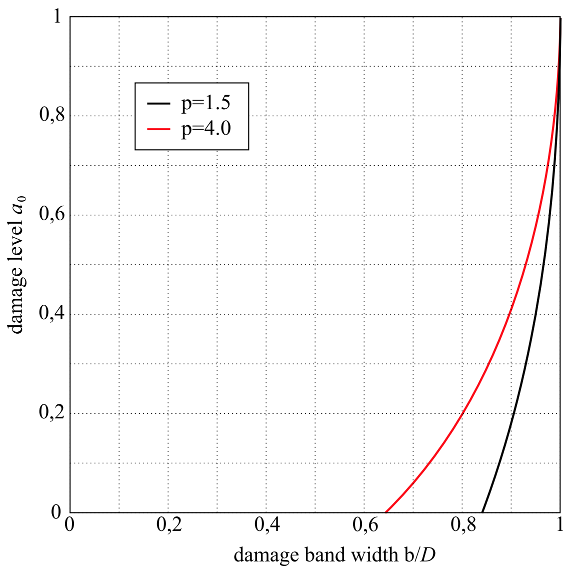

Fig. 2. Damage band width.

assumed to be centred on $x = 0$, where the damage field reaches its maximal value a_0. The width of the current damage band is denoted $b(a_0)$ so that the band spreads over the area:

$$-b(a_0) \leqslant x \leqslant b(a_0) \quad \text{with } b(a_0) = x(a_0, 0) \tag{10}$$

Note that the present closed-form solution is shown to be admissible with respect to the irreversibility condition $\dot{a} \geqslant 0$ if and only if $b(a_0)$ is increasing. At the ultimate stage $a_0 = 1$, the profile of the damage band simply reads:

$$a_u(x) = \left(1 - \frac{x}{D}\right)^2 \quad \text{with } D = \sqrt{\frac{2c}{k}} \tag{11}$$

where D denotes (half) the band width. In Fig. 1, the damage profile is plotted for two values of a_0 (0.5 and 1, see the damage axis) and two values of the internal parameter p of the stiffness function (2). The space axis x is normalised by D which brings a length scale. In that way, the ultimate damage profile (11) is constant. In Fig. 2, it can be observed how the (normalised) current damage band width $b(a_0)$ evolves with increasing damage. In particular, it is indeed an increasing function of a_0, the condition for the closed-form solution to be admissible.

Regarding the stress σ, it is constant along the bar thanks to the equilibrium. It decreases while damage progressively increases and it is related to a_0 through the following relation:

$$\sigma(a_0) = \sqrt{\frac{2Eka_0}{A(a_0)^{-1} - 1}} \tag{12}$$

In that way, the current maximal damage a_0 can indeed be considered as an (increasing) loading parameter equivalently to the (decreasing) stress σ. Moreover, $a_0 = 1$ corresponds to the ultimate stage since the stress reaches zero: the bar breaks.

Other quantities of interest may be deduced from (9) and (12). In particular, the opening displacement measured at the distance D of the centre of the damage band can be expressed as follows, with $U(a_0, x)$ the displacement at point x and corresponding to the damage level a_0:

$$\delta(a_0) \underset{def}{=} U(a_0, x = D) - U(a_0, x = -D) = \frac{2\sigma(a_0)}{E}\left[D + \int_0^{a_0} \left(A(a)^{-1} - 1\right) G(a_0, a)^{-1/2}\, \mathrm{d}a\right] \tag{13}$$

2.3. Stiffness function

The choice of the specific stiffness function (2) is inspired from the proposition in [7] for its sound numerical properties. More precisely, an asymptotic development of (13) shows that the ultimate opening displacement corresponding to $a_0 = 1$ is finite and nonzero as soon as the stiffness function reads $A(a) = (1-a)^2 R(a)^{-1}$ with R a strictly positive and continuous function over [0 1]. An exponent less than 2 would result in zero ultimate opening displacement (hence implying a snap-back) while an exponent greater than 2 would result in an infinite one. In addition, it appears that the stiffness function is involved in the closed-form solution only through the term $A(a)^{-1} - 1$:

$$A(a)^{-1} - 1 = \frac{R(a) - 1 + 2a - a^2}{(1-a)^2} \tag{14}$$

A simple choice then consists in choosing R as a polynomial of degree 2. Note that a polynomial of degree less than 2 would not fulfil the condition that the damage band width $b(a_0)$ be increasing, as will be seen in (17). As the material is initially sound, that is $A(0) = 1$, the expression (14) then reads:

$$A(a)^{-1} - 1 = \frac{ma(1 + pa)}{(1-a)^2} \tag{15}$$

with m and p the parameters already introduced above, so that it results indeed in the expression (2). Note that the parameterisation is chosen so that m appears as a factor in (9)–(13) which will prove interesting when performing an asymptotic analysis of the damage model in terms of a cohesive zone model because it will separate the nonlocal length scale from the other parameters.

Regarding its physical meaning, the function $A(a)$ should be positive and decreasing. It should also be convex, a prescription of generalised standard materials. And it should result in an elastic domain which grows in the strain space and shrinks in the stress space. It leads to $p \geqslant 0$ and $m \geqslant p + 2$. Moreover, the validity of the developments depends on the increasing character of $b(a_0)$. A numerical study shows that a sufficient condition is that $b(a_0)$ be increasing in the vicinity of $a_0 = 1$. A limited expansion of b leads to:

$$\frac{b(1-\epsilon)}{D} = 1 + \frac{p-1}{8(p+1)}\epsilon^2 \ln\epsilon + O\left(\epsilon^2\right) \tag{16}$$

In addition to the previous conditions, it implies that the present analysis is valid subjected to:

$$p \geqslant 1; \qquad m \geqslant p + 2 \tag{17}$$

3. Convergence study

3.1. Macroscopic parameters

At the present stage, the uniaxial model relies on five internal parameters: the initial stiffness E, the threshold k, the nonlocal coefficient c and the stiffness parameters m and p. The idea consists now in dividing these parameters in two sets: a length scale characteristic of nonlocality and four “macroscopic” parameters which remain constant when the length scale goes to zero. The choice of the macroscopic parameters determines the existence and the nature of the asymptotic law for a vanishing length scale, see for instance [9] which exhibits the convergence toward a Griffith fracture law.

A natural candidate for the nonlocality length scale is (half) the damage band width at ultimate failure D, as expressed in (11). Indeed, as the damage is localised in the area $-D \leqslant x \leqslant D$, a crack should correspond to the limit $D \to 0$.

Regarding the macroscopic parameters, they include the elastic stiffness E which is physically independent of the length scale. In addition, the peak stress σ_y and the fracture energy G_f are usual characteristic quantities of cohesive zone models [4]. They admit a physical meaning in the case of the gradient damage model. The peak stress is reached when damage begins ($a_0 = 0$) since the stress is a decreasing function of damage. According to (12) and (2), it reads:

$$\frac{\sigma_y^2}{2E} = \frac{k}{m} \tag{18}$$

The fracture energy can be interpreted as the energy to provide in order to break the bar. It is the sum of the stored energy (through the gradient of damage) and the dissipated energy (8) which correspond to the ultimate damage distribution a_u expressed in (11). A calculation leads to:

$$G_f = \frac{4}{3}kD \tag{19}$$

The relations (11), (18) and (19) can be inverted in order to express internal parameters as functions of the macroscopic ones and of the length scale:

$$k = \frac{3}{4}\frac{G_f}{D}; \qquad c = \frac{3}{8}DG_f; \qquad m = \frac{3}{2}\frac{EG_f}{\sigma_y^2 D} \tag{20}$$

At last, the parameter p in the stiffness function is also considered as macroscopic. The choice will prove relevant in Section 3.2 where p will be given the status of a shape parameter related in particular to the critical opening displacement (the opening displacement at failure).

In that way, the model now relies on a length scale D and four macroscopic parameters: the elastic stiffness E, the fracture energy G_f, the peak stress σ_y and the shape parameter p. Note that the condition (17) implies a restriction on the length scale:

$$D \leqslant \frac{3}{2(p+2)}\frac{EG_f}{\sigma_y^2}; \quad p \geqslant 1 \tag{21}$$

3.2. Normalisation of the closed-form solution

We will now demonstrate that the uniaxial response can be deduced from a "master response" which depends only on the shape parameter p. To highlight such a factorisation, the following convention is adopted: functions denoted with a superscript bar depend only on p and their explicit arguments. Thus, regarding the localisation band, straightforward manipulations lead to:

$$G(a_0, a) = \frac{1}{D^2}\bar{G}(a_0, a); \quad \bar{G}(a_0, a) = 4a\left[1 - \left(\frac{1-a_0}{1-a}\right)^2 \frac{1+pa}{1+pa_0}\right] \tag{22}$$

Hence the damage profile and the band width read:

$$x(a_0, a) = D\bar{x}(a_0, a); \quad \bar{x}(a_0, a) = \int_a^{a_0} \bar{G}(a_0, s)^{-1/2}\, \mathrm{d}s \tag{23}$$

$$b(a_0) = D\bar{b}(a_0); \quad \bar{b}(a_0) = \int_0^{a_0} \bar{G}(a_0, s)^{-1/2}\, \mathrm{d}s \tag{24}$$

The stress and the opening displacement admit the same treatment. It results in:

$$\sigma(a_0) = \sigma_y \bar{\sigma}(a_0); \quad \bar{\sigma}(a_0) = \frac{1-a_0}{\sqrt{1+pa_0}} \tag{25}$$

$$\delta(a_0) = \frac{G_f}{\sigma_y}\bar{\delta}(a_0) + 2D\frac{\sigma_y}{E}\bar{\sigma}(a_0); \quad \bar{\delta}(a_0) = 3\bar{\sigma}(a_0)\int_0^{a_0} \frac{a(1+pa)}{(1-a)^2}\bar{G}(a_0, a)^{-1/2}\, \mathrm{d}a \tag{26}$$

As announced, the opening displacement, the stress and the damage profile are deduced from a master response that depends only on p.

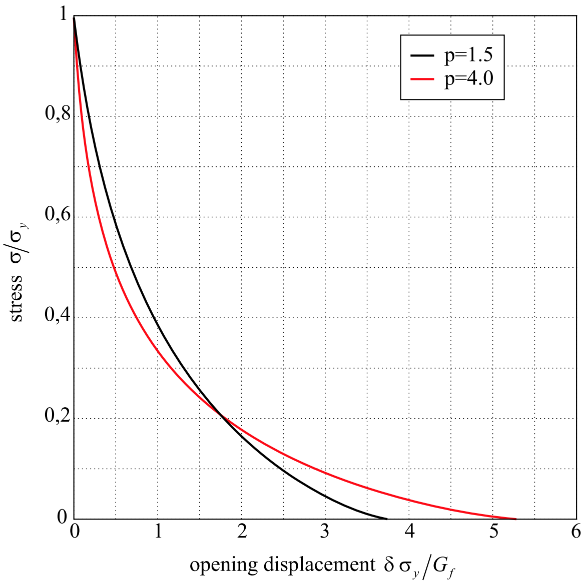

Fig. 3. Asymptotic cohesive zone model.

3.3. Consequences

It appears from (23) that the damage profile is homothetical for decreasing length scale D and hence tends to a crack when $D \to 0$. Besides, the band width variation between damage inception and failure is larger for larger values of p, since the initial and ultimate band widths are equal to:

$$\bar{b}(0) = \frac{\pi}{2\sqrt{p+2}}; \qquad \bar{b}(1) = 1 \tag{27}$$

It is also interesting to take advantage of the expression (26) in order to evaluate the asymptotic behaviour of the opening displacement at ultimate failure. After some calculations, it can be shown that this critical opening displacement $\bar{\delta}_c$ is finite (as announced) with value:

$$\lim_{a_0 \to 1} \bar{\delta}(a_0) = \frac{3\pi}{4}\sqrt{p+1} \tag{28}$$

This relation thus provides a physical interpretation for p in terms of critical opening displacement.

Finally, the expressions (25) and (26) provide the asymptotic response in terms of stress vs. opening displacement for a vanishing length scale D and constant macroscopic parameters in the 1D case. It takes the form of a cohesive zone model that obeys the following law expressed in a parametric format:

$$(\delta, \sigma) \in \left\{ \left(\frac{G_f}{\sigma_y}\bar{\delta}(s), \sigma_y \bar{\sigma}(s) \right);\ 0 \leqslant s \leqslant 1 \right\} \tag{29}$$

The corresponding (normalised) graph is plotted in Fig. 3 for two values of the shape parameter p. It highlights the properties of the cohesive law: perfect adhesion in the reversible regime, then a decrease of the stress with increasing opening displacement (no snap-back) when damage occurs and at last failure corresponding to zero stress and a finite critical opening displacement.

4. Conclusion

It has been showed that the uniaxial response of a specific gradient damage model converges toward the response of a cohesive zone model when the length scale that controls the nonlocal effects goes to zero. Even though the study is led in a one-dimensional context, it is a first step towards a more comprehensive convergence analysis. Several complementary points should be explored in that sense:

- numerical validation of the convergence in mode I in a 2D then a 3D setting;
- mathematical demonstration through Γ-convergence of the underlying energy;
- extension to mixed mode loading;
- extension to crack closure and compressive damage.

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