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Feeling the elephant[1]

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I am very honored by the election to the Academy of Sciences of France, a country with a great scientific tradition and in particular, a great mathematical tradition.

I was a student of Yuri Manin at Moscow University. I was also greatly influenced by I.I. Piatetski-Shapiro, one of the very few mathematicians in Moscow who deeply understood the Langlands program. This program mysteriously relates n-dimensional representations of the Galois group of a global field and automorphic representations of GL(n) over the adeles of the global field. ("GL(n)" denotes the group of matrices of order n.) There are two kinds of global fields: number fields (i.e., fields generated by finitely many algebraic numbers) and functional fields (i.e., fields generated by finitely many algebraic functions in one variable over a finite field). A deep analogy between these types of fields is one of the main ideas of number theory. In the number field case it was already known that to pass from automorphic representations to Galois representations one had to study the cohomology of modular curves or, more generally, of Shimura varieties. In the 1970's I introduced varieties[2] that play a similar role for functional fields and used them to construct for such fields the Galois representations corresponding to automorphic representations of GL(2) - first under an additional discrete series assumption, then without it. I was unable to treat the case of GL(n), n>2; this was done by Laurent Lafforgue in 2001.

Functional fields are in one-to-one correspondence with smooth projective curves over finite fields. To an n-dimensional representation of the fundamental group of such a curve the Langlands program associates a function on the set of rank n vector bundles on the curve. In a 1983 article I showed (for n=2) that in fact, this function comes from a finer object. Namely, it

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[1] The "elephant" includes the Langlands program but is not limited to it. The comparison with the proverbial "elephant and the 6 blind men" is due to Langlands himself.

[2] One of these varieties goes back to remarkable works of an American mathematician Leonard Carlitz, which were written in the 1930's and remained unnoticed for many years.

comes (via Grothendieck's functions-sheaves dictionary) from a complex of l-adic sheaves on the moduli space of vector bundles; moreover this complex of sheaves exists for not necessarily finite ground fields. This approach (which was later greatly developed by G. Laumon, D. Gaitsgory,and others) is now called the "geometric Langlands program". It works for any reasonable theory of sheaves and cohomology. In the particular case of de Rham theory some additional methods are available. In this case the object on the moduli space of vector bundles that one has to construct is a D-module or if you prefer, a system of linear partial differential equations. For a certain class of representations of the fundamental group Beilinson and I were able to write these differential equations explicitly using quantization of Hitchin's fibration.

In the 1980's, I worked on a project not related directly to the Langlands program. Trying to understand the works on integrable quantum systems by L.D. Faddeev and his collaborators[3] I realized that the algebraic structure behind their theory is that of Hopf algebra or, equivalently, "quantum group". Independently this was realized by Michio Jimbo. In particular, we constructed nontrivial Hopf algebra deformations of the universal enveloping algebras of simple Lie algebras.[4] A few years later, D. Kazhdan, G. Lusztig, and M. Kashiwara applied these deformations to Lie algebra representations in ways I never imagined.

Thank you for your attention.

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[3] Especially helpful was E.K.Sklyanin's thesis (J. Soviet Math., vol 19, p. 1546-1595, 1982).

[4] The simple Lie algebras themselves cannot be deformed.