INSTITUT DE FRANCE
Académie des sciences

Royaume du Maroc
Académie Hassan II

MADEV 17

Rabat, Maroc, 16-19 octobre 2017

Posters de la session *Mathématiques appliquées et contrôle des systèmes réels*

H.Zouiten & A.Boutoulout - *Boundary enlarged observability of the gradient for linear parabolic systems.*

S. Dridi, S. El Yacoubi & F. Bagnoli - *Boundary regional control of cellular automata.*

A. Mbarki & R. Oubrahim - *Cyclical k-contraction in probabilitistic metric spaces.*

S. Souhaile, L.Afifi & M. Bahadi - *Disturbances compensation for delayed hyperbolic systems .*

S. Belhedid , N. Beniich, A. El Bouhtouri , B. Abouzaid & D. Dochain - *Extremum Seeking Control for a mass structured cell population balance model in a bioreactor, and numeric simulations.*

S. Zenfari, M. Laabissi & M. E.Achhab - *Interconnection and damping assignment passivity based control for irreversible port*

Y. Hamadi, A. Omrane & D. Bouagada - *Null-Controllability for a problem in Epidemiology* SIR. *Application to the Sentinel method of Lions.*

K. Danine, S. Bernoussi & A. Bel Fekih - *Observabilité Partielle d'un Système Dynamique Linéaire.*

A. Binid, M. E. Achhab & M. Laabissi - *On positive observer for a class of infinite dimensional linear positive systems.*

M. Naim, F. Lahmidi, A. Namir & M. Rachik - *Output reachability and output controllability of positive linear discrete delay systems.*

M. Ouyadri, M. Laabissi & M.E. Achhab - *Positive state reachability of discrete time positive linear systems.*

Boundary enlarged observability of the gradient for linear parabolic systems

Hayat Zouiten & Ali Boutoulout

MACS Laboratory, Faculty of Sciences, Meknes, Morocco

E-mail:hayat.zouiten@yahoo.fr

جامعة مولاي إسماعيل UNIVERSITÉ MOULAY ISMAÏL | كلية العلوم FACULTÉ DES SCIENCES

Abstract

The aim goal of this paper is to study the notion of regional boundary observability of parabolic linear systems with constraints on the gradient. We shall explore an approach based on the Hilbert Uniqueness Method (HUM) that can reconstruct the initial gradient state between tow prescribed functions f_1 and f_2 on a part Γ of the boundary $\partial\Omega$ without the knowledge of the state. Finally, numerical results are illustrated.

Introduction

In system theory, the observability is related to the possibility of reconstruction of the initial state from the measurements taken of the system by the means of tools called sensors.

The concept of enlarged observability is a special case of the observability, which the aim is to reconstruction the initial (state or gradient) between two prescribed functions.

The are many reasons for introducing this concept :

- The mathematical model of real systems is obtained from measures or for approximations. Then, the solution for such system is approximately known and required to be only between two bounds.
- The observation error is smaller than in general case and the initial conditions to be reconstructed are to be between some constraints.
- This problem is encountered in various real problems where the reconstructed state is required only to be between two levels.

Considered systems

Let Ω be an open bounded subset of $\mathbb{R}^n$ $(n \geq 2)$, with a regular boundary $\partial\Omega$. For $T > 0$, let's consider $Q_T = \Omega\times]0,T[$ and $\Sigma_T = \partial\Omega\times]0,T[$. We consider the following parabolic system :

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t} = Ay(x,t) & in \quad Q_T \\ y(x,0) \quad = y_0(x) & in \quad \Omega \\ y(\xi,t) \quad = 0 & on \quad \Sigma_T \end{cases} \tag{1}$$

where A is a second-order linear differential operator which generates a strongly continuous semi-group $(S(t))_{t\geq 0}$ in the Hilbert space $L^2(\Omega)$. We assume that the initial state y_0 and its gradient ∇y_0 are unknown with $y_0 \in H^2(\Omega)\cap H_0^1(\Omega)$. The measurement are obtained by the output function :

$$z(t) = Cy(.,t), \quad t \in]0,T[\tag{2}$$

where $C : D(C) \subseteq H^2(\Omega)\cap H_0^1(\Omega) \longrightarrow \mathbb{R}^q$ is linear (possibly unbounded) operator and depend on the structure and the number q of the considered sensors. The observation space is $\mathcal{O} = L^2(0,T;\mathbb{R}^q)$.

Materials and Definitions

We consider the following operators :

- The observability operator

$$K : H^2(\Omega)\cap H_0^1(\Omega) \longrightarrow \mathcal{O}$$
$$z \longmapsto CS(.)z$$

- The Gradient operator

$$\nabla : H^2(\Omega)\cap H_0^1(\Omega) \longrightarrow (L^2(\Omega))^n$$
$$y \longmapsto \nabla y = \left(\frac{\partial y}{\partial x_1}, ..., \frac{\partial y}{\partial x_n}\right)$$

- The trace operator, is defined by

$$\gamma : (L^2(\Omega))^n \longrightarrow (H^{1/2}(\partial\Omega))^n$$

with $\gamma_0 : L^2(\Omega) \longrightarrow H^{1/2}(\partial\Omega)$, the trace operator of order zero.

- For $\Gamma \subseteq \partial\Omega$, the restriction operator is given by

$$\chi_\Gamma : (H^{1/2}(\partial\Omega))^n \longrightarrow (H^{1/2}(\Gamma))^n$$
$$y \longmapsto \chi_\Gamma y = y_{|\Gamma}$$

Let $(\alpha_i(.))_{i=1}^n$ and $(\beta_i(.))_{i=1}^n$ be two functions defined in $(H^{1/2}(\Gamma))^n$ such that $\alpha_i(.) \leq \beta_i(.)$ a.e. on Γ for all $1 \leq i \leq n$. Throughout the paper we set

$$[\alpha(.),\beta(.)] = \left\{(y_1,...,y_n) \in (H^{1/2}(\Gamma))^n \,|\, \alpha_i(.) \leq y_i(.) \leq \beta_i(.) \quad \text{a.e. on} \quad \Gamma \quad \forall i \in \{1,...,n\}\right\}$$

Definition 1. The system (1)-(2) is said to be $[\alpha(.),\beta(.)]$-gradient observable on Γ if

$$Im(\chi_\Gamma\gamma\nabla K^*) \cap [\alpha(.),\beta(.)] \neq \emptyset$$

Proposition 1. We have the equivalence between the following statements.

1. The system (1)-(2) is $[\alpha(.),\beta(.)]$-gradient observable on Γ.

2. $Ker(K\nabla^*\gamma^*\chi_\Gamma^*) \cap [\alpha(.),\beta(.)] = \{0\}$.

Main Problem

Can we reconstruct the initial gradient state, supposed unknowns $\nabla y_0^1 = \chi_\Gamma\gamma_0\nabla y_0$ the trace of ∇y_0 between two functions $\alpha_i(.)$ and $\beta_i(.)$ on the subregion $\Gamma \subseteq \partial\omega\cap\partial\Omega$ for all $1 \leq i \leq n$?

HUM Approach

Let $r > 0$ be arbitrary and sufficiently small, and consider

$$F_r = \bigcup_{z\in\Gamma} B(z,r) \text{ and } \omega_r = F_r \cap \Omega$$

with $B(z,r)$ is the ball of radius r centered in z. Let $(\alpha_i'(.))_{i=1}^n$ and $(\beta_i'(.))_{i=1}^n$ be two functions defined in $(L^2(\omega_r))^n$ such that $\alpha_i'(.) \leq \beta_i'(.)$ in ω_r for all $1 \leq i \leq n$, we have

$$[\alpha'(.),\beta'(.)] = \left\{(y_1,...,y_n) \in (L^2(\omega_r))^n \,|\, \alpha_i'(.) \leq y_i(.) \leq \beta_i'(.) \quad \text{a.e. in} \quad \omega_r \quad \forall i \in \{1,...,n\}\right\}$$

Then we have the result.

Proposition 2. If $\alpha_i(.)$ (respectively $\beta_i(.)$) is the restriction of the trace of $\alpha_i'(.)$ (respectively $\beta_i'(.)$) and if the system (1)-(2) is $[\alpha'(.),\beta'(.)]$- gradient observable in ω_r, then it is $[\alpha(.),\beta(.)]$-gradient observable on Γ.

Let the initial gradient state decomposed in the following form :

$$\nabla y_0 = \begin{cases} \nabla y_0^2 & \text{in } [\alpha'(.),\beta'(.)] \\ \nabla y_0^3 & \text{in } (L^2(\omega_r))^n\backslash[\alpha'(.),\beta'(.)]. \end{cases}$$

In the sequel our subject is the reconstruction of the component ∇y_0^2 between $\alpha_i'(.)$ and $\beta_i'(.)$ in ω_r and deduce its trace ∇y_0^1 between $\alpha_i(.)$ and $\beta_i(.)$ on Γ for all $1 \leq i \leq n$.

We consider the system (1) supposed observed by an internal pointwise sensor (b,δ_b), let's consider the set $\mathcal{G}$ be defined by:

$$\mathcal{G} = \{h \in (L^2(\Omega))^n \mid h = 0 \text{ in } (L^2(\omega_r))^n\backslash[\alpha'(.),\beta'(.)]\} \cap \{\nabla f \mid f \in H^2(\Omega)\cap H_0^1(\Omega)\}. \tag{3}$$

For $\theta_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, we consider the system

$$\begin{cases} \dfrac{\partial\theta(x,t)}{\partial t} = A\theta(x,t) & \text{in} \quad Q_T \\ \theta(x,0) \quad = \theta_0(x) & \text{in} \quad \Omega \\ \theta(\xi,t) \quad = 0 & \text{on} \quad \Sigma_T, \end{cases} \tag{4}$$

For $\widetilde{\theta}_0 \in \mathcal{G}$, there exists a unique $\theta_0 \in H^2(\Omega)\cap H_0^1(\Omega)$ such that $\widetilde{\theta}_0 = \nabla\theta_0$. Then we consider the semi-norm on $\mathcal{G}$ be defined by

$$\widetilde{\theta}_0 \longmapsto \|\widetilde{\theta}_0\|_{\mathcal{G}} = \left[\int_0^T \left(\sum_{k=1}^n \frac{\partial\theta}{\partial x_k}(b,t)\right)^2 dt\right]^{\frac{1}{2}}. \tag{5}$$

We consider the retrograde system :

$$\begin{cases} -\dfrac{\partial\phi(x,t)}{\partial t} = A^*\phi(x,t) + \displaystyle\sum_{k=1}^n \frac{\partial\theta}{\partial x_k}(b,t)\delta(x-b) & \text{in} \quad Q_T \\ \phi(x,T) \quad = 0 & \text{in} \quad \Omega \\ \dfrac{\partial\phi(\xi,t)}{\partial\nu_{A^*}} \quad = 0 & \text{on} \quad \Sigma_T, \end{cases} \tag{6}$$

Let the operator Λ be defined by :

$$\Lambda : \mathcal{G} \longrightarrow \mathcal{G}^*$$
$$\widetilde{\theta}_0 \longrightarrow \Lambda\widetilde{\theta}_0 = \mathcal{P}(\Phi(0)), \tag{7}$$

where $\mathcal{P}$ denoted the projection operator and $\Phi(0) = (\phi(0),...,\phi(0))$.

Let's consider the system :

$$\begin{cases} -\dfrac{\partial\psi(x,t)}{\partial t} = A^*\psi(x,t) + \displaystyle\sum_{k=1}^n \frac{\partial y}{\partial x_k}(b,t)\delta(x-b) & \text{in} \quad Q_T \\ \psi(x,T) \quad = 0 & \text{in} \quad \Omega \\ \dfrac{\partial\psi(\xi,t)}{\partial\nu_{A^*}} \quad = 0 & \text{on} \quad \Sigma_T. \end{cases} \tag{8}$$

If $\widetilde{\theta}_0$ is chosen such that $\psi(0) = \phi(0)$ in ω_r, then the system (8) could be seen as an adjoint of the system (1) together with the output (2) is and our problem of the regional boundary observability with constraints on the gradient turns up to solve the equation

$$\Lambda\widetilde{\theta}_0 = \mathcal{P}(\Psi(0)), \tag{9}$$

where $\Psi(0) = (\psi(0),...,\psi(0))$, with ψ is the solution of the system (8).

Proposition 3. If the system (1) together with the output (2) is $[\alpha'(.),\beta'(.)]$-gradient observable in ω_r, then the equation (9) admits a unique solution $\widetilde{\theta}_0 \in \mathcal{G}$, and the boundary initial gradient state to be observed between $\alpha_i(.)$ and $\beta_i(.)$ on Γ for all $1 \leq i \leq n$, is given by : $\nabla y_0^1 = \chi_\Gamma\gamma\nabla\theta_0$.

Simulation results

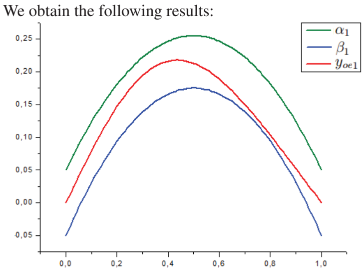
Let's consider the following two-dimensional system in $\Omega =]0,1[\times]0,1[$ excited by a pointwise sensor:

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1,x_2,t) = 0.01\left[\dfrac{\partial^2 y}{\partial x_1^2}(x_1,x_2,t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1,x_2,t)\right] & \text{in} \quad Q_T \\ y(x_1,x_2,0) \quad = y_0(x_1,x_2) & \text{in} \quad \Omega \\ y(\xi,\eta,0) \quad = 0 & \text{on} \quad \Sigma_T. \end{cases} \tag{10}$$

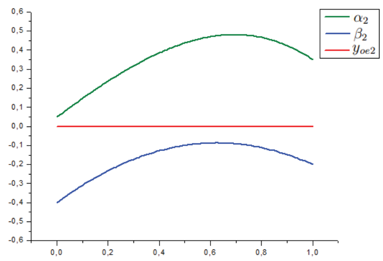
The initial gradient state to be observed on Γ be given by:

$$\begin{cases} y_1(x_1,x_2) = 2x_1x_2^2 - 2x_1x_2 - x_2^2 + x_2, \\ y_2(x_1,x_2) = 2x_1^2x_2 - 2x_1x_2 - x_1^2 + x_1. \end{cases}$$

We obtain the following results:



The first component of the estimated gradient state on Γ



The second component of the estimated gradient state on Γ

These figures show that the initial state gradient estimated is between $\alpha_i(.)$ and $\beta_i(.)$ on the subregion Γ, then the location of the sensor is $[\alpha(.),\beta(.)]$-gradient strategic on Γ. The initial gradient state is estimated with a reconstruction error $\|y_0 - y_{oe}\|^2 = 3.25\times 10^{-4}$.

Forthcoming Research

- Study the concept of enlarged observability for a class of parabolic and hyperbolic semi-linear systems.

References

[1] R.F. Curtain and H. Zwart (1995), *An Introduction to Infinite-Dimensional Linear Systems Theory*, volume 21 of Texts in Applied Mathematics. Springer-Verlag, New York.

[2] J.L. Lions (1988), *Contrôlabilité exacte perturbations et stabilisation des systèmes distribués*, Tome 1, Contrôlabilité exacte, Masson, Paris.

[3] E. Zerrik and H. Bourray , *Flux reconstruction: sensors and simulations*, Sensors and Actuators A, 109, 34-46, 2003.

Boundary Regional Control of Cellular Automata

Sara DRIDI, Samira El Yacoubi, Franco Bagnoli

Mathématiques Appliquées et contrôle des systèmes réels

Perpignan University, Florence University

sara.dridi@etudiant.univ-perp.fr, yacoubi@univ-perp.fr, franco.bagnoli@unifi.it

Abstract

In System Theory, regional controllability has been studied for distributed parameters systems described by partial differential equations. The purpose of this work is to review this concept for more general spatially extended systems. By means of Cellular Automata (CA) models, we show how we can reach a desired state when it is defined only on a given part of the domain, by acting on its boundaries. We first investigate the one-dimensional case and prove the regional controllability for linear CAs. The results which are illustrated by examples will be extended to the 2D-case. The Master/Slave synchronization method will be applied.

Keywords: *Spatially extended systems, Regional controllability, Cellular automata, Synchronization.*

1. Introduction

The System Analysis consists of the study of such concepts which allow to a better comprehension, one of the important problems is the problem of controllability which consists of the possibility of transferring the state of the system in finite time from initial state to a desired state in sub region ω of the whole domain Ω. An extension which is so important is that of the notion of Regional Controllability in which we are interested in this work.

2. Objectives

- Find the control which we need to apply on the boundaries of a linear Cellular automata to achieve the desired state for given time, such that $s_T = s_d$.
- Find the optimal control which leads the state of the system to a desired state in minimum time.

Definition 1 *We Consider a given region $\omega \in \Omega$, of positive Lebesgue measure and let p_ω the restriction map defined by:*

$$\begin{aligned} p_\omega : L^2(\Omega) &\rightarrow L^2(\omega) \\ z &\rightarrow p_\omega z = z|_\omega \end{aligned}$$

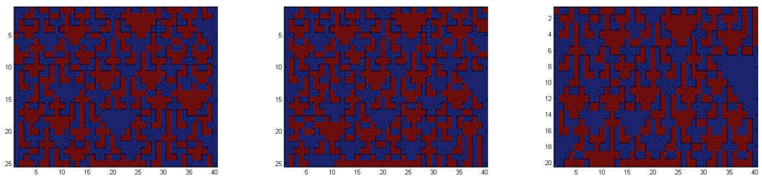
Definition 2 *The system is said to be exactly controllable on ω (or exactly $\omega-$controllable) if it exists a control $u \in L^2(0,T;\mathbb{R}^p)$ such that:* $\quad p_\omega z(T,u) = z_d$
Where z_d is the desired state.

3. Algorithm Steps

- Enter the CA dimension,
- Enter the parameters related to the linear rule,
- Initialization of the CA configuration except on the boundaries,
- Enter the time where we want to reach the desired state,
- Enter the parameters related to the region,
- Enter the desired state,
- According to the cell max position in the region ω and using the fact that each cell depends linearly on another cell in its neighbourhood at previous time, go up to search the line of cell in the boundaries which mean where we apply control u_0 on which the cell with max position in the region depend linearly,
- Depending on the line and the column of cell in control vector u_0, do the evolution of cellular automata top part which does not depend on the control choice,
- Go back on time to determine the cells states in control vector which are depending on the desired state,
- Do the evolution of cellular automata inferior part which depends on the changes in the control vector.

4. Results

Example of rule 150:
We consider 1D linear CA which is consisting of $p = 40$ cells noted c_i, $i = 1, \ldots, p$. Each cell can take a value from the set of states $\{0, 1\}$. The transition of the cell state is performed according to the neighbourhood $v(c_i) = \{c_{i-1}, c_i, c_{i+1}\}$, with the rule given by: $s_{t+1}(c_i) = s_t(c_{i-1}) \oplus s_t(c_i) \oplus s_t(c_{i+1})$. We search about the optimal control which we need to apply for getting the desired state in the region $\omega = \{c_{16}, \ldots, c_{25}\}$ such as $\forall 16 \leq i \leq 25$, $s_{T_{min}}(c_i) = 1$. We distinguish three cases:

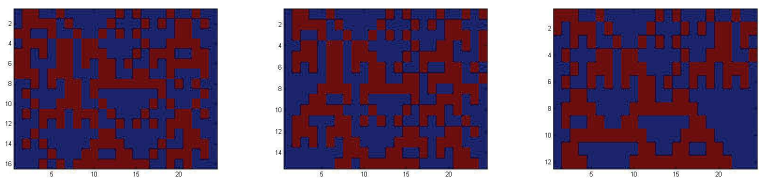


From the left | **From the right** | **From two sides**

Figure 1: ***Controllability in the case of rule 150***

Example of rule 90:

We consider 1D linear CA which is consisting of $p = 24$ cells noted c_i, $i = 1, \ldots, p$. Each cell can take a value from the set of states $\{0, 1\}$. The transition of the cell state is done according to the neighbourhood $v(c_i) = \{c_{i-1}, c_i, c_{i+1}\}$, with the rule is given by: $s_{t+1}(c_i) = s_t(c_{i-1}) \oplus s_t(c_{i+1})$. We search about the optimal control which we need to apply for getting the desired state in the region $\omega = \{c_{10}, \ldots, c_{16}\}$ such as $\forall 10 \leq i \leq 16$, $s_{T_{min}}(c_i) = 1$. We distinguish three cases:



From the left | **From the right** | **From two sides**

Figure 2: ***Controllability in the case of rule 90***

5. Conclusion

- In this work, we investigated the interesting problem of Boundary Regional Control of CA. We focussed on linear rules in the 1D case. A more exhaustive study is needed in order to find a general framework for Boundary Regional control of CA. We are currently working on the 2D case and nonlinear/chaotic CA rules.

References

[1] F. Bagnoli, S El Yacoubi and R Rechtman Control of Cellular Automata *Physical Review E*, Vol. 86, 066201-7, 2012.

[2] F. Bagnoli, S El Yacoubi and R Rechtman Chaos, Synchronization and Control in Cellular Automata, Acta Physica Polonica B Vol 5, N 1,pp, 9-19,2012.

[3] F. Bagnoli, S El Yacoubi and R Rechtman Toward a boundary Regional Control of Cellular Automata, 2016.

[4] L. Afifi, A. El Jai and E. Zerrik, Dynamical Systems: Regional Analysis of distributed systems, 2008.

[5] F. Bagnoli, S El Yacoubi, R Rechtman and Sara DRIDI Regional control and Synchronization of Cellular Automata, International Conference Chaotic Modeling, Simulation and Applications, Barcelona, Spain, 30 may- 02 june 2017.

Cyclical k-contraction in probabilistic metric spaces

Abderrahim MBARKI(1)&*Rachid OUBRAHIM*(2)
LANO Laboratory, University Mohammed First, OUJDA

Contact Information:
(1)*NATIONAL SCHOOL OF APPLIED SCIENCES, P.O.BOX669, OUJDA UNIVERSITY, MOROCCO.*
(2)*FACULTY OF SCIENCES, 60000, OUJDA UNIVERSITY, MOROCCO.*

Phone : 00212670909033
Email : dr.mbarki@gmail.com

Abstract

The purpose of this presentation is to prove a fixed point theorem for a probabilistic k-contraction restricted to two nonempty closed sets of a probabilistic metric spaces.

Introduction

Fixed points theory plays a basic role in applications of many branches of mathematics. Finding a fixed point of contractive mappings becomes the center of strong research activity.
After that, based on this finding, a large number of fixed point results have appeared in recent years. Generally speaking, there usually are two generalizations on them, one is from spaces, the other is from mappings.
Concretely, for one thing, from spaces, for example, the concept of a probabilistic metric spaces was introduced in 1942 by Karl Menger [1], indeed, he proposed replacing the distance $d(p,q)$ by a real function F_{pq} whose value $F_{pq}(x)$ for any real number x is interpreted as the probability that the distance between p and q is less than x.
For another thing, from mappings, for instance, let A and B be nonempty subsets of a metric space (M,d) and let $f : A\cup B \to A\cup B$ be a mapping such that:
(1) $f(A) \subseteq B$ and $f(B) \subseteq A$.
(2) $d(fx, fy) \le kd(x,y)$, $\forall x \in A$, $\forall y \in B$, where $k \in [0,1)$.
If (1) holds we say that f is a cyclic map and if (1) and (2) hold we say that f is a cyclic contraction [2].

1 Preliminaries

Definition 1.1. *A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F defined on* $^+\cup\{\infty\}$ *that satisfies* $f(0) = 0$ *and* $f(\infty) = 1$, *and is left continuous on* $(0,\infty)$. *The set of all d.d.f's will be noted by* Δ^+; *and the set of all F in* Δ^+ *for which* $\lim_{t\to\infty} f(t) = 1$ *by* D^+.

For any a in $\mathbb{R}^+\cup\{\infty\}$, ε_a, the unit step at a, is the function given by:
for $0 \le a < \infty$

$$\varepsilon_a(x) = \begin{cases} 0 & \text{if } 0 \le x \le a \\ 1 & \text{if } a < x \le \infty \end{cases}$$

and

$$\varepsilon_\infty(x) = \begin{cases} 0 & \text{if } 0 \le x \le \infty \\ 1 & \text{if } x = \infty \end{cases}$$

Note that $\varepsilon_a \le \varepsilon_b$ if and only if $b \le a$; that ε_a is in D^+ if $0 \le a < \infty$; and that ϵ_0 is the maximal element, and ϵ_∞ the minimal element, of Δ^+.

Definition 1.2. *Consider* f *and* g *be in* Δ^+, $h \in (0,1]$, *and let* $(f,g;h)$ *denotes the condition*

$$0 \le g(x) \le f(x+h) + h,$$

for all x *in* $(0, \frac{1}{h})$.
The modified Levy distance is the function d_L *defined on* $\Delta^+\times\Delta^+$ *by*
$d_L(f,g) = inf\{h :$ *both conditions* $(f,g;h)$ *and* $(g,f;h)$ *hold*$\}$.
Note that for any f *and* g *in* Δ^+, *both* $(f,g;1)$ *and* $(g,f;1)$ *hold, hence* d_L *is well-defined and* $d_L(f,g) \le 1$.

Lemma 1.1. *The function* d_L *is a metric on* Δ^+.

Definition 1.3. *A sequence* $\{F_n\}$ *of d.d.f's is said to converge weakly to a d.d.f. F if and only if the sequence* $\{F_n(x)\}$ *converges to* $F(x)$ *at each continuity point* x *of* F.

Lemma 1.2. *Let* $\{F_n\}$ *be a sequence of functions in* Δ^+, *and let F be in* Δ^+. *Then* $\{F_n\}$ *converges weakly to F if and only if* $d_L(F_n, F) \to 0$.

Lemma 1.3. *The metric spaces* (Δ^+, d_L) *is compact, and hence complete.*

Lemma 1.4. *For any F in* Δ^+ *and* $t > 0$,

$$F(t) > 1 - t \ iff \ d_L(F, \varepsilon_0 < t).$$

Lemma 1.5. *If F and G are in* Δ^+ *and* $F \le G$ *then* $d_L(G,\varepsilon_0) \le d_L(F,\varepsilon_0)$.

Definition 1.4. *A triangular norm (briefly, a t-norm) is a binary operation T on* $[0,1]$ *such that:*

$$\begin{aligned} T(x,y) &= T(y,x), \ (commutativity) \\ T(x,y) &\le T(z,w), \ whenever \ x \le z, \ y \le w, \\ T(x,1) &= x, \ (1 \ is \ an \ identity \ element) \\ T(T(x,y),z) &= T(x,T(y,z)), \ (associativity). \end{aligned}$$

Example 1.1. *The following t-norms are continuous:*
(i) *The t-norm minimum* $M(x,y) = Min(x,y)$.
(ii) *The t-norm product* $\prod(x,y) = xy$.
(iii) *The t-norm* W, $W(x,y) = Max(x+y-1, 0)$.

Definition 1.5. *A triangle function is a binary operation* τ *on* Δ^+ *that is commutative, associative, and nondecreasing in each place, and has* ε_0 *as identity.*

Example 1.2. *If T is left continuous, then the binary operation* τ_T *on* Δ^+ *defined by:*

$$\tau_T(F,G)(x) = \sup\{T(F(u), G(v)) : u + v = x\},$$

is a triangle function.

Lemma 1.6. *If T is continuous, then* τ_T *is continuous .*

Definition 1.6. *A probabilistic metric space (briefly a bms) is a triple* (M,F,τ) *where M is a nonempty set, F is a function from* $M\times M$ *into* Δ^+, τ *is a triangle function, and the following conditions are satisfied*
for all $p, r; q \in S$
(i) $F_{pp} = \varepsilon_0$,
(ii) $F_{pr} = \varepsilon_0 \implies p = r$,
(iii) $F_{pr} = F_{rp}$
(iv) $F_{pr} \ge \tau(F_{pq}, F_{qr})$.
If $\tau = \tau_T$ *for some t-norm T, then* (M,F,τ_T) *is called a Menger space.*

It should be noted that if T is a continuous t-norm, then (M,F) satisfies (iv) under τ_T if and only if it satisfies

$$(v) \ \ F_{pr}(x+y) \ge T(F_{pq}(x), F_{qr}(y)),$$

for all $p, r; q \in M$ and for all $x, y > 0$, under T.

Definition 1.7. *Let* (M,F) *be a probabilistic semimetric space (i.e.,* (i), (ii) *and* (iii) *of Definition 2.6 are satisfied). For p in M and* $t > 0$, *the strong t-neighborhood of p is the set*

$$N_p(t) = \{q \in M : \ F_{pq}(t) > 1 - t\}.$$

The strong neighborhood system at p is the collection $\wp_p = \{N_p(t) : \ t > 0\}$, *and the strong neighborhood system for M is the union* $\wp = \bigcup_{p\in M}\wp_p$.

An immediate consequence of Lemma 2.4 is

$$N_p(t) = \{q \in M : d_L(F_{pq}, \varepsilon_0) < t\}.$$

In probabilistic semimetric space, the convergence of sequence is defined in the way

Definition 1.8. *Let* $\{x_n\}$ *be a sequence in a probabilistic semimetric space* (M,F). *Then*
(1) The sequence $\{x_n\}$ *is said to be convergent to* $x \in M$, *if for every* $\epsilon > 0$, *there exists a positive integer* $N(\epsilon)$ *such that* $F_{x_n x}(\epsilon) > 1-\epsilon$ *whenever* $n \ge N(\epsilon)$.
(2) The sequence $\{x_n\}$ *is called a Cauchy sequence, if for every* $\epsilon > 0$ *there exists a positive integer* $N(\epsilon)$ *such that* $n, \ m \ge N(\epsilon) \Rightarrow F_{x_n x_m}(\epsilon) > 1-\epsilon$.
(3) (M,F) *is said to be complete if every Cauchy sequence has a limit.*

Proposition 1.1. *Let* $\{x_n\}$ *be a sequence in a probabilistic semimetric space* (M,F) *and* $x \in M$.
$1-$ $\{x_n\}$ *is convergent to x, if either*
- $\lim_{n\to\infty} F_{x_n x}(t) = 1$ *for all* $t > 0$, *or*
- *for every* $\epsilon > 0$ *and* $\delta \in (0,1)$, *there exists a positive integer* $N(\epsilon,\delta)$ *such that* $F_{x_n x}(\epsilon) > 1-\delta$, *whenever* $n \ge N(\epsilon,\delta)$.
$2-$ $\{x_n\}$ *is Cauchy sequence, if either*
- $\lim_{n,m\to\infty} F_{x_n x_m}(t) = 1$ *for all* $t > 0$, *or*
- *for every* $\epsilon > 0$ *and* $\delta \in (0,1)$, *there exists a positive integer* $N(\epsilon,\delta)$ *such that* $F_{x_n x_m}(\epsilon) > 1-\delta$, *whenever* $n, \ m \ge N(\epsilon,\delta)$.

Scheizer and Sklar [3] proved that if (M,F,τ) is a probabilistic metric space with τ is continuous, then the family I consisting of $\emptyset$ and all unions of elements of this strong neighborhood system for M determines a Hausdorff topology for M.
Consequently, in such space we have the following assertions
(a) (M,F,τ) is endowed with the topology I is a Hausdorff topological space.
(b) There exists a topology Λ on S such that the strong neighborhood system $\wp$ is a basis for Λ.

Let f a self map on M. Power of f at $p \in M$ are defined by $f^0 p = p$ and $f^{n+1}p = f(f^n p)$, $n \ge 0$. We will use the notation $p_n = f^n p$, in particular $p_0 = p$, $p_1 = fp$.
The letter Ψ denotes the set of all function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$0 < \varphi(t) < t \ \ and \ \ lim_{n\to\infty}\varphi^n(t) = 0 \ \ for \ \ each \ \ t > 0$$

Definition 1.9. *[4] We say that a t-norm T is of H-type if the family* $\{T^n(t)\}$ *is equicontinuous at* $t = 1$, *that is,*
$\forall\epsilon \in (0,1), \exists\lambda \in (0,1) : t > 1-\lambda \Rightarrow T^n(t) > 1-\epsilon, \ \ \forall n \ge 1$
Where $T^1(x) = T(x,x)$, $T^n(x) = T(x, T^{n-1}(x))$, *for every* $n \ge 2$.

The t-norm T_M is a trivial example of t-norm of H-type.

Definition 1.10. *[5] Let* $\varphi : [0,\infty) \to [0,\infty)$ *be a function such that* $\varphi(t) < t$ *for* $t > 0$, *and f be a selfmap of a probabilistic metric space* (M,F,τ). *We say that f is* φ*-probabilistic contraction if*

$$F_{fpfq}(\varphi(t)) \ge F_{pq}(t).$$

for all $p,q \in M$ *and* $t > 0$,

Theorem 1.1. *[6] Let* (M,F,τ_T) *be a complete probabilistic metric space under a continuous t-norm T of H-type such that* $RanF \subset D^+$. *Let* $f : M \to M$ *be a* φ*-probabilistic contraction where* $\varphi \in \Psi$. *Then f has a unique fixed point* $\overline{x}$, *and, for any* $x \in M$, $\lim_{n\to\infty} f^n(x) = \overline{x}$.

2 Cyclical contractive conditions in probabilistic metric spaces

Theorem 2.1. *Let* $(M,F,\tau_T,)$ *be a complete probabilistic metric space under a continuous t-norm T of H-type such that* $RanF \subset D^+$. *Let* $f : M \to M$ *be a continuous mapping and satisfies*

$$F_{fpf^2p}(kt) \ge F_{pfp}(t).$$

for all $p \in M$ *and* $t > 0$ *where* $k \in (0,1)$.
Then f has a fixed point in M.

Theorem 2.2. *Let* $(M,F,\tau_T,)$ *be a complete probabilistic metric space under a continuous t-norm T of H-type such that* $RanF \subset D^+$. *Let A and B be nonempty closed subsets of M and let* $f : A\cup B \to A\cup B$ *be a mapping and satisfies:*
(1) $F(A) \subset B$ *and* $F(B) \subset A$.
(2) $F_{fpfq}(kt) \ge F_{pq}(t)$, $\forall p \in A$ *and* $\forall q \in B$, *where* $k \in (0,1)$.
Then f has a unique fixed point in $A \cap B$.

References

[1] K. Menger, Statistical metrics, Proc. Natl. Acad. Sci. **28** (1942), 535-537.

[2] W. A. Kirk, P. S. Srinivasan, P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions. Fixed Point Theory, volume 4, No. **1**, 2003, 79-89.

[3] Schweizer B. and A.Sklar, Probabilistic Metric Spaces, North-Holland Series in Probability and Applied Mathimatics, **5**, (1983).

[4] O. Hadzić, A fixed point theorem in Menger spaces, Publ. Inst. Math. (Beograd) T **20** (1979) 107–112.

[5] A. Mbarki, A. Benbrik, A. Ouahab, W. Ahid and T. Ismail, Comments on "Fixed Point Theorems for φ-Contraction in Probabilistic Metric Space", Int. J. Math. Anal. **7**, 2013, no. 13, 625 - 635.

[6] L. Ciric, Solving the Banach fixed point principle for nonlinear contractions in probabilistic metric spaces, Nonlinear Analysis **72** (2010) 2009-2018.

Mathématiques Appliquées et contrôle

Disturbances compensation for delayed hyperbolic systems

Souhaile Salma* & Larbi Afifi* & Bahadi Mohamed**

*MACS Laboratory, Faculty of Sciences, University Hassan II Ain Chock.
**MACS Laboratory, Ecole Royale Navale.

souhaile.salma@gmail.com
larbi.afifi@gmai.com
mohamedbahadi@yahoo.fr

Abstract

In this paper, we examine the remediability problem for a class of hyperbolic systems with an input delay. The problem consists to compensate the disturbance effect in finite time T. We demonstrate how to find the optimal control ensuring the exact remediability of a known or unknown disturbance. We give the main properties and characterization results for such systems according to the delay concept.

Introduction

The delay phenomenon and its characteristics are natural and appear obviously in various domains such as biology, economics and population dynamics. The modeling of time delay process involves the past informations (history) for predicting the future behavior of the system. Delay systems or also called hereditary systems, belong to the class of functional differential equation with an infinite dimension.
This paper concerns the problem of exact remediability for a class of delayed hyperbolic systems. The considered problem consists to find an input operator B ensuring the compensation of any disturbance f (generally detected through the observation), i.e. bringing at the final time T, the output equation to the normal observation (with $f = 0$). This problem was explored for various types of systems without delay (parabolic or hyperbolic, distributed or lumped, discrete or continuous) and also for the parabolic case with constant or time-variant delay.
We introduce at first, the problem statement of the hyperbolic system and its reformulation structure to a linear model, then we illustrate the resolution prototype using the semigroups theory. We examine thereafter the corresponding compensation problem with numerical results in the case of one dimension wave equation.

Main Objectives

1. Linearize the hyperbolic equation (wave equation).
2. Resolve the corresponding system, i.e. to give the output result of the linearized system.
3. Define the corresponding compensation problem (or the exact remediability concept).
4. Find the input operator ensuring the compensation problem.
5. Prove the results (example and numerical results).

Problem statement and considered system

We consider the system described by the following equation:

$$\begin{cases} \dfrac{\partial^2 x}{\partial t^2}(\xi,t) = \Delta x(\xi,t) + Bu(t-h) + f(t) & \Omega\times]0,T[\\ x(\xi,0)=0; \quad \dfrac{\partial x}{\partial t}(\xi,0)=0 & \Omega \\ x(\eta,t)=0 & \partial\Omega\times]0,T[\\ u(\alpha)=\phi(\alpha) & \alpha\in[-h,0], \quad h\geq 0 \end{cases} \tag{1}$$

augmented with the output equation:

$$y(t) = \begin{pmatrix} C_1 x(.,t) \\ C_2 \dfrac{\partial x}{\partial t}(.,t) \end{pmatrix} \tag{2}$$

where $C_1 \in \mathcal{L}(L^2(\Omega), Y_1)$ and $C_2 \in \mathcal{L}(L^2(\Omega), Y_2)$. Let A be the operator defined by $A\psi = \Delta\psi$ for $\psi \in D(A) = H^2(\Omega)\bigcap H_0^1(\Omega)$, and $z = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$ on $Z = H_0^1(\Omega)\times L^2(0,T;\Omega)$, with the inner product:

$$< \psi, \psi' > = < (-A)^{1/2}\psi_1, (-A)^{1/2}\psi_1' >_\Omega + < \psi_2, \psi_2' >_\Omega$$

for $\psi =< \psi_1, \psi_2 >\in Z$. The system (1) is equivalent to:

$$\begin{cases} \dot{z}(t) = \mathcal{A}z(t) + \mathcal{B}u_t + \mathcal{F}(t); & 0<t<T \\ z(0)=0 \end{cases} \tag{3}$$

where $\mathcal{F} = \begin{pmatrix} 0 \\ f \end{pmatrix} \in Z$, $\mathcal{B} = \begin{pmatrix} 0 \\ B \end{pmatrix} \in \mathcal{L}(U,Z)$ with $U = U_1\times U_2$ and the input restriction $u_t \in L^2(-h,0;\mathrm{U})$ is given by:

$$u_t(\alpha) = u(t+\alpha); \qquad \alpha\in[-h,0] \tag{4}$$

The operator $\mathcal{A}$ is defined by:

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \tag{5}$$

with $D(\mathcal{A}) = D(A)\times H_0^1(\Omega)$. $\mathcal{A}$ generates a strongly semigroup defined by (see [1] and [3]):

$$S(t)\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \displaystyle\sum_{n=1}^{+\infty}\sum_{j=1}^{r_n}[< z_1, \varphi_{nj} >_{L^2(\Omega)} \cos(\sqrt{-\lambda_n}t) \\ + \dfrac{1}{\sqrt{-\lambda_n}} < z_2, \varphi_{nj} >_{L^2(\Omega)} \sin(\sqrt{-\lambda_n}t)]\varphi_{nj} \\ \\ \displaystyle\sum_{n=1}^{+\infty}\sum_{j=1}^{r_n}[-\sqrt{-\lambda_n} < z_1, \varphi_{nj} >_{L^2(\Omega)} \sin(\sqrt{-\lambda_n}t) \\ + < z_2, \varphi_{nj} >_{L^2(\Omega)} \cos(\sqrt{-\lambda_n}t)]\varphi_{nj} \end{pmatrix} \tag{6}$$

where $(\varphi_{nj})_{j=1,\dots,r_n}$ is a complete orthogonal system of eigenfunctions of A, associated to the eigenvalues $(\lambda_n)_{n\geq1}$ such that $0 > \lambda_1 > \lambda_2 > \dots\lambda_n$; r_n is the multiplicity of λ_n.
To express the restriction input, we consider the left-shift semigroup describing the input solution (for more details, one can see [2], [4] and [5]):

$$\begin{cases} Q & := \dfrac{\partial}{\partial\alpha} \\ D(Q) & := \phi\in W^{1,2}(-h,0;\mathrm{U}) : \phi(0)=0 \end{cases}$$

where ϕ define the input history. $(Q, D(Q))$ generates a strongly continuous semigroup:

$$(\Psi(t)\phi)(\alpha) = \begin{cases} 0 & t+\alpha\geq 0 \\ \phi(t+\alpha) & t+\alpha\leq 0 \end{cases} \quad \alpha\in[-h,0] \tag{7}$$

We define the linear operator:

$$(\Phi(t)u)(\alpha) = \begin{cases} u(t+\alpha) & t+\alpha\geq 0 \\ 0 & t+\alpha\leq 0 \end{cases} \quad \alpha\in[-h,0] \tag{8}$$

The input control u_t introduced in equation (4) is given by (see [6]):

$$u_t(\alpha) = (\Psi(t)\phi)(\alpha) + (\Phi(t)u)(\alpha)$$

At the final time T sufficiently large, the output, also noted $y_{u,f}$, defined as follows:

$$y(T) = \begin{pmatrix} \displaystyle\sum_{n=1}^{+\infty}\sum_{j=1}^{r_n}\int_0^T\left[\frac{1}{\sqrt{-\lambda_n}} < B\Phi(s)u + f(s), \varphi_{nj} > \sin(\sqrt{-\lambda_n}(T-s))\right] ds C_1\varphi_{nj} \\ \displaystyle\sum_{n=1}^{+\infty}\sum_{j=1}^{r_n}\int_0^T\left[< B\Phi(s)u + f(s), \varphi_{nj} > \cos(\sqrt{-\lambda_n}(T-s))\right] ds C_2\varphi_{nj} \end{pmatrix} \tag{9}$$

In this case, we define the remediability problem as follows:
For $\mathcal{F}\in L^2(0,T;Z)$ does a control $u\in L^2(0,T;U)$ ensuring at the final time T, the compensation of the effect of $\mathcal{F}$ on the observation, i.e.: $y(T)=0$. If u exists, is it optimal?

Compensation problem

The problem now consists to find the minimum energy control ensuring the exact remediability, i.e.:

$$\min_{v\in\mathcal{C}} J(v) = \|u\|^2_{L^2(0,T;U)} \tag{10}$$

where

$$J(u) = \|u\|^2_{L^2(0,T;U)} = \left(\int_0^T \|u(t)\|^2_U\right)^{\frac{1}{2}}$$

and

$$\mathcal{C} = \{u\in L^2(0,T;U),\ u \text{ satisfies } y(T)=0\}$$

then, for $\mathcal{F}\in L^2(0,T;\mathrm{Z})$, does a control $u\in L^2(0,T;\mathrm{U})$ such $y(T)=0$ is verified? If u exists, is it optimal? i.e. u is the solution of (10).
Under the weak remediability hypothesis, we show that the optimal control is given by:

$$\begin{aligned} u^{\theta_f}(s) = \sum_{n=1}^{+\infty}\sum_{j=1}^{r_n}[\sqrt{-\lambda_n} < C_1^*\theta_{f_1}, \varphi_{nj} > \sin(\sqrt{-\lambda_n}(T-s-\alpha)) \\ - < C_2^*\theta_{f_2}, \varphi_{nj} > \cos(\sqrt{-\lambda_n}(T-s-\alpha))]\ B^*\varphi_{nj} \end{aligned} \tag{11}$$

where $\theta_f = \begin{pmatrix} \theta_{f_1} \\ \theta_{f_2} \end{pmatrix} = C^*(CHH^*C^*)^{-1}\left(\int_0^T CS(T-r)f(r)dr\right)$ and the delay $\alpha = -h$.
The usual case of actuators and sensors is also examined. Indeed, if the corresponding system is excited by p actuators $(\Omega_i, g_i)_{1\leq i\leq p}$, the control space is defined by $U=\mathbb{R}^p$ and the operator B is given by

$$\begin{aligned} B:\ \mathbb{R}^p &\rightarrow \quad L^2(\Omega) \\ u(t) &\rightarrow \sum_{i=1}^{p} g_i u_i(t-h) \end{aligned} \tag{12}$$

where $u(t) = (u_1(t-h),\dots,u_p(t-h))^{tr}\in L^2(0,T;\mathbb{R}^p)$ with the constant delay h. The adjoint B^* of B is then given, for all $x\in L^2(\Omega)$, by:

$$B^*x = (< g_1, x >, \dots, < g_p, x >)^{tr}, \quad \mathcal{B}^* = \begin{pmatrix} 0 & B^* \end{pmatrix}$$

Now we consider the input restriction $u_t\in L^2(-h,0;\mathbb{R}^p)$ defined as follows:

$$u_t = (u_{t,1}, u_{t,2}, \dots, u_{t,p})^{tr}$$

where

$$u_{t,i}(\alpha) = u_i(t+\alpha); \quad \alpha\in[-h,0]$$

then Bu_t can be written:

$$Bu_t = \sum_{i=1}^{p} g_i u_{t,i}$$

Now, if the output of the system is given by q_1 and q_2 sensors $(D_i, r_i)_{1\leq i\leq q_1}$ and $(D_i', k_i)_{1\leq i\leq q_2}$, we have respectively:

$C_1 z_1 = (< r_1, z_1 >, ..., < r_{q_1}, z_1 >)^{tr}$ and $C_2 z_2 = (< k_1, z_2 >, ..., < k_{q_2}, z_2 >)^{tr}$. Then at the final time T, the output is given by:

$$y(T) = \begin{pmatrix} \sum_{n=1}^{+\infty} \sum_{j=1}^{r_n} \int_0^T \left[\frac{1}{\sqrt{-\lambda_n}} < \sum_{i=1}^{p} g_i u_i(T-h) + f(s), \varphi_{nj} > \sin(\sqrt{-\lambda_n}(T-s)) \right] ds C_1 \varphi_{nj} \\ \sum_{n=1}^{+\infty} \sum_{j=1}^{r_n} \int_0^T \left[< \sum_{i=1}^{p} g_i u_i(T-h) + f(s), \varphi_{nj} > \cos(\sqrt{-\lambda_n}(T-s)) \right] ds C_2 \varphi_{nj} \end{pmatrix}$$

Here also, we examine the existence of an optimal control ensuring the compensation of a disturbance f, i.e. $y(T) = 0$. It is given by:

$$\begin{aligned} u^{\theta_f}(s) = & \sum_{n=1}^{+\infty} \sum_{j=1}^{r_n} [\sqrt{-\lambda_n} < C_1^* \theta_{f_1}, \varphi_{nj} > \sin(\sqrt{-\lambda_n}(T-s+h)) \\ & - < C_2^* \theta_{f_2}, \varphi_{nj} > \cos(\sqrt{-\lambda_n}(T-s+h))] \; B^* \varphi_{nj} \end{aligned} \tag{13}$$

Application and numerical simulations

As an application, we consider the case where $\Omega =]0,1[$ and the system is excited by a one actuator (Ω, g). We assume that the observation is given by two sensors (D, r) and (D', k); i.e. $y(t) = Cz(t)$ or $y_{u,f} = \begin{pmatrix} y_{1,u,f} \\ y_{2,u,f} \end{pmatrix}$ is the observation corresponding to the control u and the disturbance f. Hence $y_{1,u,f} = \langle r, x(.,t) \rangle$ and $y_{2,u,f} = \langle k, \frac{\partial x}{\partial t}(.,t) \rangle$, and the corresponding optimal control is given by:

$$\begin{aligned} u^{\theta_f}(s) = \sum_{n \geq 1} \Big(& n\pi \langle r, \varphi_n \rangle \langle g, \varphi_n \rangle \sin(n\pi(T-s-h)) \theta_{f_1} \\ & + \langle k, \varphi_n \rangle \langle g, \varphi_n \rangle \cos(n\pi(T-s-h)) \theta_{f_2} \Big) \end{aligned} \tag{14}$$

where $\varphi_n(.) = \sqrt{2} \sin(n\pi.)$. In the particular situation where $r = k = g = \varphi_1$, we have:

$$u^{\theta_f}(s) = \sin(\pi(T-s-h)) \theta_{f_1} + \cos(n\pi(T-s-h)) \theta_{f_2} \tag{15}$$

We give hereafter respectively the two component y_1 and y_2 of the observation y with $f = \exp(t)$:

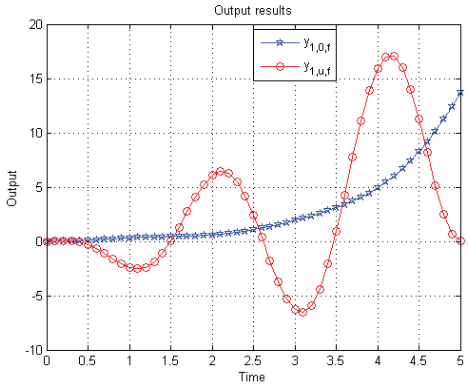


Figure 1: y_1 for $T = 5$

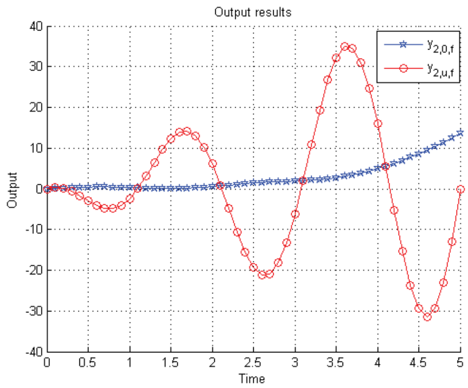


Figure 2: y_2 for $T = 5$

Similarly, we observe the output where $f(t) = \sqrt{t}$ and $T = 5$:

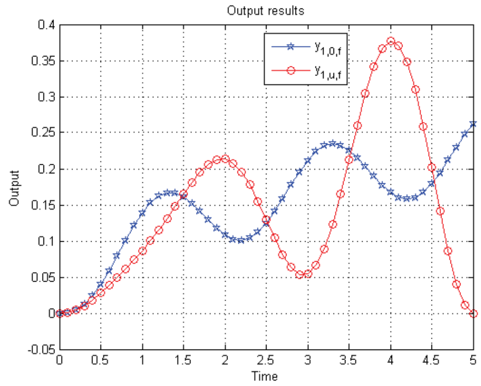


Figure 3: y_1 for $T = 5$

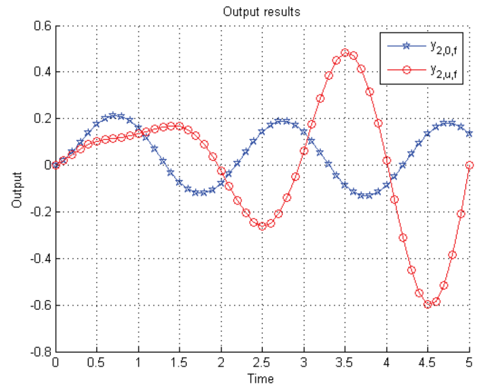


Figure 4: y_2 for $T = 5$

Conclusion

This paper is an extension of the remediability concept in the case of delayed input. The purpose is to study the possibility to find a convenient input operator (actuators), with respect to the output one (sensors) ensuring the compensation of any (or a class of) disturbance(s) at the final time T. Therefore, the main properties and characterization results are presented and examined in the usual situation of a one dimension wave equation. We also show how to obtain the optimal control (minimum energy) ensuring the compensation (exact remediability) of a disturbance f . These results depend on the hyperbolic aspect of the considered system and naturally on the applied delay operator.
To conclude, the compensation problem for a class of dynamical delay systems with disturbance is a contemporary target in scientific research. Various works are developed, but many other complex systems are still arousing further investigation.

References

[1] L. Afifi and A. El Jai. *Systèmes distribués perturbés*, volume 1. Presses Universitaires de Perpignan, 2015.

[2] András Bátkai and Susanna Piazzera. *Semigroups for delay equations*. AK Peters Wellesley, 2005.

[3] R. F. Curtain and A.J. Pritchard. *Infinite dimensional linear systems theory*. Springer-Verlag New York, 1978.

[4] O. Diekmann, S.A. Gils, S. M.V. Lunel, and H.Walther. *Delay Equations*. Springer-Verlag New York, 1995.

[5] K.J. Engel and R Nagel. *One-parameter semigroups for linear evolution equations*. Springer, New York, 2000.

[6] S. Hadd and Qing-Chang Zhong. On Feedback Stabilizability of Linear Systems With State and Input Delays in Banach Spaces. *IEEE Transactions on Automatic Control*, 54(3):438–451, March 2009.

Faculté des Sciences El Jadida

Extremum Seeking Control for a mass structured cell population balance model in a bioreactor

[1]BELHDID Salaheddine, [1]BENIICH Nadia, [1]EL BOUHTOURI Abdelmoula
[2]ABOUZAID Boouchra & [3]DENIS Dochain

[1]Laboratoire de Mathématiques Appliquées, Equipe d'ingénierie Mathématiques INMA Faculté des sciences d'El Jadida, Maroc
[2]Ecole Nationale des Sciences Appliquées d'El Jadida, Université Chouaib Doukkali, Maroc
[3]CESAME, Université Catholique de Louvain, Belgium

Abstract

In this paper, we present an adaptive Extremum seeking control scheme for a mass structured cell population balance model in a bioreactor. We assume limited knowledge of the reaction kinetics and the objective function. An adaptive Extremum seeking control is designed to steer the system states to the equilibrium number of cells of the reaction mixture. Lyapunov's stability theorem is used in the design of the Extremum seeking controller structure. Under mild assumptions the resulting controller steer the optimum of an objective function.

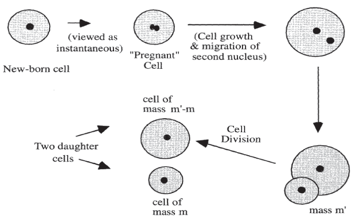
Key words: Extremum Seeking, Lyapunov function, Adaptive control, Nonlinear Systems.

Introduction

The majority of control schemes is focused on regulation and tracking of system states to known set points or trajectories. The objective of Extremum seeking is to seek the operating set points that maximize or minimize an objective function, add to this that Extremum-seeking control is a class of adaptive control that deals with regulation to unknown set-points. The controller seek the operating set-points that optimize a performance function. The uncertainty associated with the function makes it necessary to use some sort of adaptation to search for the optimal operating conditions. Many successful applications of Extremum control approaches have seen light in the literature. In this study, we investigate an Extremum seeking scheme for a mass structured cell population balance model in a bioreactor. Only limited knowledge of the reaction kinetics are assumed. A Lyapunov-based adaptive control technique is used to approximate the unknown kinetics and to steer the system to its unknown Extremum.The used technique ensures convergence of the system to an adjustable neighbourhood of its unknown optimum.

Model description

Schematic of model of cell growth and division



A cell population consists of individual cells. Each cell of the population undergoes the so-called cell cycle, during which it grows and after a certain point, it divides and partitions its cellular material into two daughter cells, each of which enters its own cell

cycle.

Let us consider a cell population growing in bioreactor. The cells are distinguishable from each other in terms of their mass or any other property of the cell, which obeys the conservation law.

The cell

$$\frac{\partial N(m,t)}{\partial t} = -\frac{\partial}{\partial m}[r(m,S)N(m,t)] - \Gamma(m,S)N(m,t) - DN(m,t) + 2\int_0^1 \Gamma(m',S)p(m,m')N(m',t)dm'$$

$$\frac{dS}{dt} = D(S_f - S) - \frac{1}{Y}\int_0^1 r(m,S)N(m,t)dm.$$

With the initials and bondaries conditions

$$N(m,0) = N_0 \;;\; r(1,S)N(1,t) = r(0,S)N(1,t) = 0 \;;\; S(0) = S_0$$

Such that:
$N(m,t)$ is the number of cells with mass between m and $m+dm$ at t.
$r(m,S)$ the growth rate.
$m \in [0,1]$ the mass of cell.
$\Gamma(m,S)$ the division rate of cells.
S the concentration of the substance.
S_f the concentration of the substance at the entrance.

And let us define the zeroth moment

$$M_0 = \int_0^1 N(m,t)\ dm$$

Such that

$$\dot{M}_0 = -DM_0(t) + \int_0^1 \Gamma(m,S)N(m,t)\,dm$$

Extremum Seeking control problem

we consider the extremum seeking problem for:

$$\begin{cases} \dot{M}_0(t) & = -DM_0(t) + \int_0^1 \Gamma(m,S)N(m,t)dm' \\ \dot{S} & = D(S_f - S) - \frac{1}{Y}\int_0^1 (m,S)N(m,t)dm \end{cases} \tag{2.1}$$

Such that:
$N(m,t)$ is the number of cells with mass between m and $m+dm$ at t.
$r(m,S)$ the growth rate.
$m \in [0,1]$ the mass of cell.
$\Gamma(m,S)$ the division rate of cells.
S the concentration of the substance.
S_f the concentration of the substance at the entrance.
M_0 the zeroth moment defined as follow:

$$M_0 = \int_0^1 N(m,t)dm$$

with the objective function

$$y = HM_0 \qquad H \in \mathbb{R}^{1*n}$$

and

$$u = DS_f > 0$$

u is the controller to design.
Our contribution was to design an adaptive Extremum seeking control that maximise the objective function. In other words that maximise the total number of cells M_0.

Design algorithm

- System equilibria.
- Impose some assumptions.
- Approximate the objective function in the equilibrium, using a Neural Network approximation technique (radial basis functions).
- Estimate the unknown parameters in the approximation.
- Estimate the gradient of the objective function, with respect to the substract concentration S.
- Using Lyapunov stability, a controler is designed to bring the process, to points where the gradient vanishes, and where the second order derivatives of y is negative.

Results and Conclusions

We consider the Lyapunov function

$$V = \frac{1}{2}\eta_1^2 + \frac{1}{2}\eta_2^2 \qquad \text{such that} \qquad \begin{cases} \eta_1 = e_S - c_1(t)^T\tilde{W} \\ \eta_2 = z_S - c_2(t)^T\tilde{W} \end{cases}$$

$d(t) \in C^1$ a dither signal. And

$$z_S = \hat{W}_p^T dZ(S) - d(t)$$

With the two dynamics

$$\begin{cases} \dot{c}_1(t)^T = -k_S c_1(t)^T + F(S) \\ \dot{c}_2(t)^T = -k_z c_2(t)^T + \Gamma_2 F(S) \end{cases}$$

Such that k_S *and* k_z two positifs gains.

$$\Gamma_2 = \hat{W}_p^T d^2 Z(S)$$

And

$$F(S) = [DZ(S)^T, DH^T Z(S)^T]$$

The designed control is

$$u = DS - F(S)W^* - k_d d(t)$$

And k_d a parameter to choose.

Simulation results

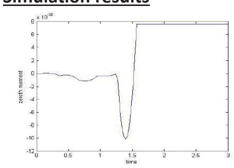


FIGURE 1. objective function

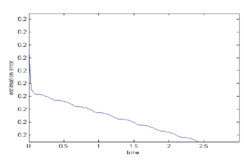


FIGURE 2. estimation error

- Under some mild assumptions we succeeded to design an adaptive Extremum Seeking control that seek the optimum of the objective function, in this case the maximum number of cells produced in this bioreactor. the proposed adaptive Extremum seeking controller guarantees the exponential convergence of the production rate of the bioreactor to an adjustable neighborhood of its maximum. And convergence of errors to an adjustable neighborhood of zero.

Interconnection and damping assignment passivity based control for Irreversible Port Hamiltonian Systems (IPHS)

[1]S. Zenfari, [2]M. Laabissi and [3]M. E. Achhab

Département de mathématiques Faculté des sciences Université Chouaib Doukkali, BP 20, El Jadida

[1]saida.zenfari1991@gmail.com [2]mlaabissi@yahoo.fr [3]elarbi.achhab@gmail.com

Abstract : In this poster we address the problem of stabilization of Irreversible Port Hamiltonian Systems (IPHS) by the so called Interconnection and Damping Assignment Passivity Based Control (IDA-PBC). In particular we define a new IDA-PBC design procedure adapted with the special structure of these IPHS, and we derive conditions on a matrix named the dissipation matrix such that the system is stabilizable via energy balancing.

Introduction

In this work we are interested in the problem of stabilization of irreversible port hamiltonian systems via the generalization of IDA-PBC technique, first proposed in [1] for systems represented in port hamiltonian form, and extensively studied in [2] for port hamiltonian systems where the control acts through the interconnection structure.

The basic idea of IDA-PBC for port hamiltonian systems [1, 2] is to control their behavior by assigning a desired closed loop structure in the following sence : Firstly, an interconnection and damping matrices are selected, after that a PDE parametrized by the chosen matrices is derived. Finally from the family of solutions of the obtained PDE an energy function is selected in such a way it satisfies the minimum requirement, and it will be used to compute the static state-feedback control.

Main Objectives

1. Extend the IDA-PBC method to IPHS.
2. Look for conditions under which the energy function of the closed loop system is at the same time constant and equals to the difference between the stored and supplied energy.
3. Stabilize the system at non zero equilibrium.

Background

The input affine representation of IPHS is defined by the dynamic equation and output relation :

$$\begin{aligned} \dot{x} &= R(x,\frac{\partial U}{\partial x},\frac{\partial S}{\partial x})J\frac{\partial U}{\partial x}+g(x,\frac{\partial U}{\partial x})u(t), \\ y &= g^T(x,\frac{\partial U}{\partial x})\frac{\partial U}{\partial x}(x) \end{aligned} \tag{1}$$

where :

1. $x(t)\in\mathbb{R}^n$ is the state vector,
2. $u(t)\in\mathbb{R}^m$ is an input a time dependant function, $g(x,\frac{\partial U}{\partial x})\in\mathbb{R}^m$,
3. $U(x)\in\mathbb{R}$,$S(x)\in\mathbb{R}$ represent respectively the internal energy (the hamiltonian) and the entropy functions,
4. $J\in\mathbb{R}^n\times\mathbb{R}^n$ is a skew symmetric matrix, structure matrix of the Poisson bracket $\{.,.\}_J$, where $\{S,U\}_J=\frac{\partial S^T}{\partial x}(x)J\frac{\partial U}{\partial x}(x)$.
5. $R=R(x,\frac{\partial U}{\partial x},\frac{\partial S}{\partial x})$ is the product of a positive definite function γ and the Poisson bracket of S and U.

$$R(x,\frac{\partial U}{\partial x},\frac{\partial S}{\partial x})=\gamma(x,\frac{\partial U}{\partial x})\{S,U\}_J \tag{2}$$

with $\gamma(x,\frac{\partial U}{\partial x}):\mathbb{R}^n\times\mathbb{R}^n\longrightarrow\mathbb{R}^n$, $\gamma\geq 0$, a non linear positive function.

By construction IPHS satisfies the first and second principles of thermodynamic which are given respectively by :

$$\frac{dU}{dt}=y^Tu, \tag{3}$$

$$\begin{aligned} \frac{dS}{dt} &= R(x,\frac{\partial U}{\partial x},\frac{\partial S}{\partial x})\frac{\partial S^T}{\partial x}J\frac{\partial U}{\partial x}+\frac{\partial S^T}{\partial x}g(x,\frac{\partial U}{\partial x})u(t), \\ &= \gamma(x,\frac{\partial U}{\partial x})\{S,U\}_J^2+(g^T(x,\frac{\partial U}{\partial x})\frac{\partial S}{\partial x})^Tu \end{aligned} \tag{4}$$

where $\gamma(x,\frac{\partial U}{\partial x})\{S,U\}_J^2=\sigma(x,\frac{\partial U}{\partial x})\geq 0$.

The system (1) is said to be stabilizable via energy balancing if

$$\int_0^t u^T(s)y(s)ds=U(x(t))-U(x(0)),\qquad \forall t\geq 0. \tag{5}$$

Main result

Our main objective in this section is to find a control $u=\beta(x)$ such that the closed loop dynamics is an IPHS of the form

$$\dot{x}=(-\sigma M+RJ_d)\frac{\partial U_d}{\partial x}(x), \tag{6}$$

where $U_d(x)$ has a strict local minimum at the desired equilibrium x_*, $J_d(x)=-J_d^T(x)$ and $M(x)=M^T(x)\geq 0$ are respectively the desired interconnection and dissipation matrices.

Controller design

Proposition 1. *Consider the IPHS* (1)*, and let $x_*\in\mathbb{R}^n$ be the desired equilibrium to be stabilized. Assume there exist functions $\beta(x)$, $J_a(x)$, $M(x)$ and a vector function $K(x)$ satisfying*

$$(-\sigma M+R(J+J_a))K(x)=-(-\sigma M+RJ_a)\frac{\partial U}{\partial x}(x)+g(x,\frac{\partial U}{\partial x})u(t) \tag{7}$$

and such that

1.

$$\begin{aligned} J_d(x) &:= J(x)+J_a(x), && (8)\\ M(x) &:= M^T(x)\geq 0. && (9)\end{aligned}$$

2. $K(x)$ is the gradient of a scalar function. That is

$$\frac{\partial K}{\partial x}(x)=[\frac{\partial K}{\partial x}(x)]^T. \tag{10}$$

3. (Equilibrium assignment) $K(x)$ at x_* satisfy

$$K(x_*)=-\frac{\partial U}{\partial x}(x_*). \tag{11}$$

4. The jacobian of $K(x)$, at x_*, verifies

$$\frac{\partial K}{\partial x}(x_*)>-\frac{\partial^2 U}{\partial x^2}(x_*). \tag{12}$$

Then the closed loop system is an IPHS of the form (6)*, where*

$$U_d(x):=U(x)+U_a(x) \tag{13}$$

and

$$\frac{\partial U_a}{\partial x}(x)=K(x). \tag{14}$$

Furthermore, x_* will be a (locally) stable equilibrium of the closed loop. If in addition, the largest invariant set under the closed loop dynamics contained in

$$\{x\in\mathbb{R}^n:[\frac{\partial U_d}{\partial x}(x)]^TM(x)\frac{\partial U_d}{\partial x}(x)=0\} \tag{15}$$

equals $\{x_*\}$, x_* will be asymptotically stable.

Remark 1. *– The target system* (6) *is chosen in such a way that the irreversible port hamiltonian nature and structure of the original system are preserved.*

Necessary condition for energy balancing stabilization

Proposition 2. *Consider the IDA-PBC of the last proposition* (1)*. Assume there exist a dissipation matrix $M(x)=M^T(x)\geq 0$ and a skew symmetric matrix $J_a(x)=-J_a^T(x)$. Then the IDA-PBC technique of proposition* (1) *is an energy balancing if*

$$M(x)\frac{\partial U}{\partial x}(x)=0,\qquad M(x)\frac{\partial U_a}{\partial x}(x)=0 \tag{16}$$

Remark 2. *The construction of the matrices M, J_a may be done as follows : Firstly, for a given U_d, choose the matrix function $M(x)$ being orthogonal on $\frac{\partial U}{\partial x}=0$, and after that solve the matching equation* (7) *to get J_a. The condition $M(x)\frac{\partial U_a}{\partial x}=0$ will be used to verify our computation.*

Sufficient condition for energy balancing stabilization

Consider the nonlinear system

$$\begin{cases} \dot{x} = f(x,\frac{\partial U}{\partial x},\frac{\partial S}{\partial x})+g(x,\frac{\partial U}{\partial x})u \\ y = h(x,\frac{\partial U}{\partial x}) \end{cases} \tag{17}$$

Proposition 3. *Assume* (17) *is a passive systems with differentiable storage function $U(x)$. Let x_* be an admissible equilibrium of* (17)*. If there exist a vector function $\beta(x)$ such that*

– The PDE

$$[f(x,\frac{\partial U}{\partial x},\frac{\partial S}{\partial x})+g(x,\frac{\partial U}{\partial x})\beta(x)]^T=-(g(x,\frac{\partial U}{\partial x})\beta(x))^T\frac{\partial U}{\partial x}, \tag{18}$$

is solvable for $U_a(x)$.

– The function $U_d(x)$ has a minimum at x_*. Then, the control law $u=\beta(x)$ is an energy balancing stabilizer for the equilibrium x_*.

Conclusions

We have defined for IPHS an IDA-PBC methodology that takes on account the structural thermodynamical properties of these IPHS. The construction is done in such a way that the irreversible hamiltonian structure is preserved, indeed the target system is also an IPHS, which facilitates for us the design and give us more freedom in the choice of the control law.

Forthcoming Research

See if the universal stabilizer property of IDA-PBC, that is IDA-PBC generates all asymptotically stabilizer controller for IPHS, is preserved for IPHS.

Références

[1] Ortega, R., van der Schaft, A., Mareels, I. & Maschke, B. (2001). Putting energy back in control. IEEE Control Systems Magazine, 21(2),18-33.

[2] Ortega, R., van der Schaft, A., Maschke, B. & Escobar, G. (2002). Interconnection and damping assignment passivity based control of port-controlled Hamiltonian systems. Automatica, 38, 585-596.

[3] Ramirez, H., Le Gorrec, Y. & Maschke, B., Couenne, F. (2016). On the passivity based control of irreversible processes : A port-Hamiltonian approach. Automatica, 64, 105-111.

[4] Ramirez, H., Maschke, B. & Sbarbaro, D.(2012). Irreversible port-Hamiltonian systems : A general formulation of irreversible processes with application to the CSTR. Chemical Engineering Science, 89, 223-234.

Null-Controllability for a problem in Epidemiology SIR Application to the Sentinel Method of Lions

UNIVERSITÉ ABDELHAMID IBN BADIS MOSTAGANEM
Yassamina HAMADI, Abdennebi OMRANE and Djilali BOUAGADA
hamadi.yasmine@yahoo.fr, Omrane@gmail.com and djillali.bouagada@univ-mosta.dz
Mathématiques Appliquées et Contrôle des Systèmes Réels
ENPO

Motivation

* New approach for determining the null controllability conditions of a perturbed problem in Epidemiology.

* The use of the Lions method to find null controllability for a problem in Epidemiology SIR.

* Interesting properties of the Lions method.

Objectives

* Application of this new approach to academic and real examples to find the null controllability.

* Comparison to state-of-the-art methods.

Preliminaries

Definition 1 *We say that S is defined by*

$$\mathcal{S}(\lambda,\tau)=\int_Q (h_0\chi_O+v\chi_\omega)\,N(\lambda,\tau)dxdt \qquad (1)$$

Definition 2 *We say that S is defined by the sentinel h, O, and ω if there is a control v such that the pair (v, S) satisfies :*

$$\begin{aligned} \frac{\partial\mathcal{S}}{\partial\tau}(0,0) &= 0, \qquad \forall\ \widehat{N}^o\in L^2(\Omega) \qquad (2)\\ \|v\|_{L^2(\omega\times(0,T))} &= \min_{u\epsilon L^2(\omega\times(0,T))}\|u\| \qquad (3)\end{aligned}$$

Proposition 1 *Let q be the solution of the adjoint problem then the problem of existence of a sentinel insensitive to the missing term is equivalent to a controllability problem, that is to say*

$$q(0)=0, \quad in\ \Omega. \qquad (4)$$

Main problem

Let us consider the following system that models the state of the infected population :

$$\begin{cases} \partial_t N-\delta\Delta N &= \lambda\widehat{I} & \text{in } Q,\\ N(0) &= \tau\widehat{S^0}+I^0+R^0 & \text{in } \Omega,\\ N &= 0 & \text{on } \Sigma\end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $Q=\Omega\times[0,T]$, $T>0$ large enough, and $\Sigma=\partial\Omega\times]0,T[$.

S, I and R represent the number of people that they involve coupled equations relating the number of susceptible people, the number of people infected, and the number of people who have recovered respectively, and $N(t)=S(t)+I(t)+R(t)$.

Main results

Proposition 2 *Let be $\overline{\rho}\in V$, the unique solution . Then the couple $(\overline{v},\overline{q})$ given by*

$$\overline{v}=-\overline{\rho}\chi_\omega \qquad (5)$$

and by

$$\overline{q}=L\overline{\rho} \qquad (6)$$

is the unique solution to the null-controllability

We suppose that the density of population N is observed on O and ω with observation :

$$N_{\text{observed}}=m_0.$$

$$\int_Q N_\lambda(h_0\chi_O+v\chi_\omega)=\int_Q(h_0\chi_O+v\chi_\omega)(m_0N_0)dxdt=\langle q,\lambda\widehat{I}\rangle$$

which contains the information on $\lambda\widehat{I}$.

References

[1] O. Bodart and P. Demeestere. Sentinels for the identification of an unknown boundary. Mathematical Models and Methods in Applie Sciences. Vol.7, N6 (1997) 871-885.
[2] H. Brezis. Analyse fonctionnelle, Collection mathématiques appliquées pour la maîtrise, Masson, 1984.
[3] P. Demeestere. Sentinel method: comparative study and application to the identification of a boundary control of the explosion time of a combustion equation. Doctoral thesis. University of Technology of Compiegne 1997.
[4] E. Fernandez-Cara and S. Guerrero. Global Carleman inequalities for parabolic systems and applications to controllability, SIAM J. Control Optim. 45 (2006) 1395-1446.
[5] A. V. Fursikov and O.Yu Imanuvilov. Controllability of evolution equations. Lecture Notes, Research Institute of Mathematics, Seoul National University, Korea, (1996).
[6] W. O. Kermack and A. G. Mckendick. A Contribution to the Mathematical Theory of Epidemics. Proc. Roy. Soc. Lond. A, 115, 700-721 (1927).
[7] J.P. Kernevez. The Sentinel Method and Its Application to Environment Pollution Problems CRC Press, New York, 1997.
[8] J.L. Lions. Sentinelles pour les systèmes distribués C.R Acad.Sci Paris Ser I.t.307,pp 819-823 (1988).
[9] J.L. Lions. Sentinelles pour les systèmes distribués à données incomplètes, Masson Paris, (1992).
[10] J.L. Lions and E. Magenes. Problèmes aux limites non homogènes et applications.Paris, Dunod, volumes 1 et 2 (1968).
[11] M.J.Keeling and P. Rohani. Modeling Infectious Diseases in Humans and animals. University Press Princeton 384 pp 2008.
[12] J.D. Murray. Mathematical biology (2nd edition) Biomathematics Texts, 19 springer.
[13] R.R. Ross. The prevention of malaria 2nd edition Londres.

Observabilité Partielle d'un Système Dynamique Linéaire

Med. El. Danine, A. Bernoussi, A. Bel fekih

Laboratoire de Géoinformation et Aménagement du Territoire (GAT)
Equipes de Modélisation Mathématique et Contrôle (MMC)
Faculté des Sciences et Techniques - Université Abdelmalek Essaadi - Tanger, Maroc
mohamed.elkhalil.danine,a.samed.bernoussi,abelfekih,(@gmail.com)

Résumé

Dans ce travail, et pour des systèmes de dimensions finies non nécessairement observables, une nouvelle approche est développée pour généraliser la notion d'observabilité. La définition d'un sous espace observable est donnée, des caractérisations sont établies et des résultats sont prouvés. Une reconstruction partielle de l'état est présentée.

1. Introduction

L'analyse d'un système dynamique utilise plusieurs concepts dont le concept d'observabilité. L'étude des systèmes observables a été faite et a déduit des caractérisations de cette classe de systèmes. Mais pour les systèmes non observables la question restait ouverte.

Un système de dimension finie non observable n'est sûrement pas "totalement non observable", il serait observable sur certains sous espaces au sens d'une certaine définition qui généralise celle de l'observabilité à donner. Nous donnons ci-dessous une telle généralisation.

2. Système considéré

Soit $A \in \mathcal{M}(n,n)$, le système considéré est décrit par les deux équations suivantes :

$$(S)\ \begin{cases} \dot{z}(t) &= Az(t), \quad \forall t \in]0,T[\\ z(0) &= z_0 \end{cases}$$

$$(E)\ \ y(t) = Cz(t), \quad \forall t \in]0,T[.$$

avec $C \in \mathcal{M}(n,p)$ et $T > 0$.

Notons $\mathcal{K}$ un opérateur défini par :

$$(\mathcal{K}z_0)(t) = CS_t z_0 \quad , \quad \text{pour tout } 0 \leq t \leq T.$$

Notons aussi $\mathbf{M} = \mathcal{K}^*\mathcal{K}$.

3. Définitions et propriétés

Définition 1 *On dit que le système observé $(S)+(E)$ est observable sur H pendant l'intervalle de temps $[t_0,T]$ si deux états du systèmes sur $[t_0,T]$ qui donnent une même sortie ont la même projection sur H pendant $[t_0,T]$:*

$$(\ y(t) = \widetilde{y}(t),\ \forall t \in [t_0,T]\ ) \Longrightarrow (\ \mathrm{P}_H z(t) = \mathrm{P}_H \widetilde{z}(t),\ \forall t \in [t_0,T]$$

on dira alors que H est observable sur $[t_0,T]$.

Pour un sous espace H de $\mathbb{R}^n$ introduisons les matrices $\mathbf{G}_H$ carrée d'ordre n et $\mathbf{Q}_H$ de type $n^2 \times n$:

$$\mathbf{G}_H = \int_{t_0}^{T} e^{(t-t_0)A^{\mathrm{T}}} (\mathrm{P}_H)^{\mathrm{T}} \mathrm{P}_H e^{(t-t_0)A} \mathrm{d}t \quad , \quad \mathbf{Q}_H = \begin{bmatrix} \mathrm{P}_H \\ \mathrm{P}_H A \\ \vdots \\ \mathrm{P}_H A^{n-1} \end{bmatrix} \tag{1}$$

On sait que le système est observable sur $H = \mathbb{R}^n$ (ou totalement obsérvable) si, et seulement si, la matrice d'observabilité :

$$\mathbf{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad \text{(de type } q \times n\text{)}$$

est de rang n

Proposition 1 *Les propositions suivantes sont équivalentes :*

1. *H est observable ;*
2. $$\mathit{Ker}(\mathbf{M}) \subseteq \bigcap_{t_0 \leqslant t \leqslant T} \mathit{Ker}\left(\mathrm{P}_H e^{(t-t_0)A}\right) \tag{2}$$
3. $$\bigcap_{k=1}^{n} \mathit{Ker}\left(CA^{k-1}\right) \subseteq \bigcap_{k=1}^{n} \mathit{Ker}\left(\mathrm{P}_H A^{k-1}\right); \tag{3}$$
4. $$\mathit{Ker}(\mathbf{M}) \subseteq \mathit{Ker}(\mathbf{G}_H); \tag{4}$$
5. $$\mathit{Ker}(\mathbf{O}) \subseteq \mathit{Ker}(\mathbf{Q}_H). \tag{5}$$

Proposition 2 *Soient H_1 et H_2 deux sous espaces vectoriels de $\mathbb{R}^n$.*

1. *Si $H_1 \subseteq H_2$ et H_2 observable alors H_1 est observable ;*
2. *Si H_1 ou H_2 est observable alors $H_1 \cap H_2$ est observable ;*
3. *Si H_1 et H_2 sont observables alors leur somme $H_1 + H_2$ est aussi observable.*

Proposition 3
- *Le sous espace $\mathrm{Im}(\mathbf{M})$ est observable. C'est le plus grand sous espace observable ; Il contient tous les sous espaces observables.*
- *Tout sous espace H de $\mathbb{R}^n$ est une somme directe orthogonale $H = H_0 \oplus H_1$ avec H_0 observable et H_1 non observable et ne contenant aucun sous espace observable. Cette décomposition est unique.*

4. Reconstruction des projections de l'état

Définition 2 *Soit y^{mes} une mesure obtenue sure $]t_0,T[$. On appelle espace des états initiaux correspondants à la mesure y^{mes} l'ensemble de tous les états initiaux qui peuvent donner cette mesure :*

$$X_{\mathrm{mes}} = \{z_0 \in \mathbb{R}^n \ /\ \mathcal{K}z_0 = y^{\mathrm{mes}}\} \tag{6}$$

Proposition 4 *Les vecteurs de X_{mes} sont caractérisés par l'équation normale*

$$\mathbf{M}z_0 = \mathcal{K}^* y^{\mathrm{mes}}$$

Proposition 5 *1. Pour un vecteur quelconque $z_0 \in X_{\mathrm{mes}}$ fixé, on a*

$$X_{\mathrm{mes}} = z_0 + \mathit{Ker}(\mathbf{M}) \tag{7}$$

2. X_{mes} est un fermé convexe de $\mathbb{R}^n$;
3. Les état initiaux de X_{mes} engendrent des états du système dont les projections coincident sur tout sous espace vectoriel observable H :

$$\forall z_0, \widetilde{z}_0 \in X_{\mathrm{mes}}\ :\ \mathrm{P}_H e^{(t-t_0)A} z_0 = \mathrm{P}_H e^{(t-t_0)A} \widetilde{z}_0 \qquad \textit{sur } [t_0,T] \tag{8}$$

Définition 3 (et notations)
- *On appelle partie visible de l'état initial du système correspondant à la mesure y^{mes} sur $[t_0,T]$, et on note $z^\star(t_0)$, l'élément de X_{mes} qui minimise la norme sur X_{mes} :*

$$z^\star(t_0) \in X_{\mathrm{mes}} \quad \textit{et} \quad \|z^\star(t_0)\| \leqslant \|z_0\|, \quad \forall z_0 \in X_{\mathrm{mes}} \tag{9}$$

- *On note par $z^\star(t)$ l'état du système engendré par la partie visible de l'état initial :*

$$z^\star(t) = e^{(t-t_0)A} z^\star(t_0) \quad , \quad t \geqslant t_0$$

Théorème 1 *1. Nous avons*

$$\mathrm{Im}(\mathbf{M}) \cap X_{\mathrm{mes}} = \{\ z^\star(t_0)\ \} \tag{10}$$

2. $z^\star(t_0)$ coincide avec la projection orthogonale de $z(t_0)$ sur $\mathrm{Im}(\mathbf{M})$;

3. Tous les vecteurs de X_{mes} ont $z^\star(t_0)$ pour projection orthogonale sur $\mathrm{Im}(\mathbf{M})$:

$$\forall z_0 \in X_{\mathrm{mes}}\ :\ \mathrm{P}_{\mathrm{Im}(\mathbf{M})}(z_0) = z^\star(t_0)$$

Proposition 6 *La partie visible de l'état initial admet pour expression*

$$z^\star(t_0) = \left[\int_{t_0}^{T} e^{(\tau-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} C e^{(\tau-t_0)A} \mathrm{d}\tau\right]^{+} \int_{t_0}^{T} e^{(s-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} y^{\mathrm{mes}}(s)\,\mathrm{d}s \tag{11}$$

Notons $\lambda_1, \ldots, \lambda_r$ les valeurs propres non nulles, distincts ou confondues de $\mathbf{M}$, qu'on ordonne

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_r > 0$$

(où $r = \mathrm{rg}(\mathbf{M}) = \mathrm{rg}(\mathbf{O})$).
On sait que $\mathbf{M}$ se décompose

$$\mathbf{M} = UDU^T \tag{12}$$

avec $U, D \in \mathbf{M}_n(\mathbb{R})$, U orthogonale ($UU^T = I_n$) et D diagonale de la forme

$$D = \begin{bmatrix} \Lambda & \mathrm{O} \\ \mathrm{O} & \mathrm{O}_{n-r} \end{bmatrix} \quad \text{où} \quad \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} \in \mathcal{M}_r(\mathbb{R}) \tag{13}$$

Posons

$$U = \begin{bmatrix} Q & L \\ G & R \end{bmatrix}, \quad z^\star(t_0) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathcal{K}^* y^{\mathrm{mes}} = \begin{bmatrix} u \\ v \end{bmatrix} \tag{14}$$

avec

$$Q \in \mathcal{M}_r(\mathbb{R}) \quad , R \in \mathcal{M}_{n-r}(\mathbb{R}) \ , G \in \mathcal{M}_{n-r,r}(\mathbb{R})$$

$$L \in \mathcal{M}_{r,n-r}(\mathbb{R}) \ , \quad x_1, u \in \mathbb{R}^r \quad , \quad x_2, v \in \mathbb{R}^{n-r}$$

Proposition 7 *La partie visible de l'état initial peut être calculé par*

$$z^\star(t_0) = \begin{bmatrix} Qw \\ Gw \end{bmatrix} \ \textit{avec}\ w = \Lambda^{-1}\left[Q^T u + G^T v\right]$$

où Q, G sont les matrices apparaissant dans les décompositions (12)(13)(14) de la matrice $\mathbf{M}$ et u, v dans $\mathbb{R}^r \times \mathbb{R}^{n-r}$ du vecteur

$$\int_{t_0}^{T} e^{(s-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} y^{\mathrm{mes}}(s)\,\mathrm{d}s$$

Théorème 2 *La projection orthogonale de l'état sur un sous espace observable H se détermine, dans l'ordre, par :*
1.

$$z^\star(t_0) = \left[\int_{t_0}^{T} e^{(\tau-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} C e^{(\tau-t_0)A} \mathrm{d}\tau\right]^{+} \int_{t_0}^{T} e^{(s-t_0)A^{\mathrm{T}}} C^{\mathrm{T}} y^{\mathrm{mes}}(s)\,\mathrm{d}s$$

où B^+ désigne le pseudo inverse de B.
2.

$$z^\star(t) = e^{(t-t_0)A} z^\star(t_0) \quad , \quad t \geqslant t_0$$

3.

$$\mathrm{P}_H z(t) = \mathrm{P}_H z^\star(t) \quad , \quad t \geqslant t_0$$

Corollaire 1 *Si l'état du système est localisé dans un sous espace observable H à un instant $t \in [t_0,T]$, alors il est donné par*

$$z(t) = \mathrm{P}_H z^\star(t)$$

Théorème 3 *La partie visible de l'état se détermine par*

$$z^{\mathrm{v}}(t) = \mathrm{P}_{\mathrm{Im}(\mathbf{M})} z^\star(t) \quad , \quad t \geqslant t_0$$

et la projection orthogonale de l'état sur un sous espace observable H se détermine par

$$\mathrm{P}_H z(t) = \mathrm{P}_H z^{\mathrm{v}}(t) = \mathrm{P}_H z^\star(t) \quad , \quad t \geqslant t_0$$

5. Conclusion et Perspectives

L'application la plus importante de la notion d'observabilité partielle est le fait qu'on peut observer une partie de l'état du système même si le système n'est pas totalement observable ce qui permet dans beaucoup de cas d'avoir une bonne idée sur l'éta du système.

Comme perspectives on peut étudier le cas des systèmes à paramètres distribués ou même des systèmes non linéaires.

Références

[1] El Jai, A., Amouroux. M. and Zerrik E., *Regional observability of distributed systems.* 301-313, International Journal of Systems Science. 2007.

[2] EL JAI, A. and PRITCHARD. A. *Sensors and Controls in the Analysis of Distributed Systems.* Series in Applied Mathematics (New York : Wiley). 1988.

[3] Ben-Israel, Adi and Greville, Thomas NE. *Generalized inverses : theory and applications*, Volume 15. Springer Science & Business Media. 2003.

Ce travail est réalisé dans le cadre du projet : PPR2/2016/79, OGI-Env, Soutenu par le MENFPESRS et le CNRST et le Réseau International Théorie des Systèmes (TDS) soutenu par l'Académie Hassan II des Sciences et Techniques.

جامعة شعيب الدكالي

Université Chouaïb Doukkali

On positive observer for a class of infinite dimensional linear positive systems

Abdellaziz. Binid, M. Elarbi Achhab, Mohamed Laabissi

Département de Mathématiques Faculté des Sciences, Université Chouaib Doukkali BP 20, El-Jadida, 24000, Maroc

Abstract

In this work, we investigate the question of designing a positive Luenberger like observer for a class of infinite-dimensional linear positive systems. The existence of such positive observers is proved by specific choice of the observer gain and using positive bounded perturbation results. We show also that the error of the observer converges exponentially to zero. Finally, the main result is applied to an isothermal tubular (bio) reactor model, namely plug-flow (bio) reactor model. The approach is illustrated by some numerical simulations.

1. Motivation

Positive linear systems are linear dynamical systems whose states and outputs are nonnegative whenever the initial condition and input control are nonnegative. Such positive systems appear in several fields, as the positivity property occurs quite frequently in practical applications where the state variables correspond to quantities that do not have real meaning unless they are nonnegative, e.g. populations, concentrations, temperatures, cell birth etc.

Objective: Design a positive observer for a class of infinite-dimensional linear positive systems.

The state of the system is not accessible $+$ positive $\Longrightarrow$ positive state estimation (observer)

2. Preliminaries

Let us consider the linear infinite dimensional system described by the following equation:

$$\begin{cases} \dot{x}(t) = Ax(t) \quad x(0) = x_0 \in H \\ y(t) = Cx(t) \end{cases} \tag{1}$$

Where

- A is the infinitesimal generator of a C_0-semi group $(T_A(t))_{t\geq 0}$ on H that satisfies

$$\forall \ t \geq 0 \qquad \| T_A(t) \| \leq Me^{-\omega t} \tag{2}$$

- $C \in \mathcal{L}(H, Y)$ the output operator;
- $(H, \|\cdot\|)$ is Banach lattice space with positive cone $H^+ = \{x \in H \mid 0 \leq x\}$;

The system (1) is said to be positive if $\forall x_0 \in H^+$ if the corresponding trajectory $x(t) \in H^+$ and the output $y(t) \in Y^+$ for all $t \geq 0$.

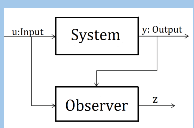


Figure 1: *Observer design*

Our aim is to construct a positive observer for the system given by (1) of the following form:

$$\begin{cases} \dot{z}(t) = (A + LC)z(t) + LCx(t) \\ z(0) = z_0 \in H \end{cases} \tag{3}$$

Where $L \in \mathcal{L}(Y, H)$ is the observer gain.

The system given by (3) is a positive observer if the error $e(t) = x(t) - z(t)$ converges to zero and (3) is a positive system. i.e $(T_{A+LC}(t))_{t\geq 0}$ and LC are positive.

The error $e(t)$ satisfies the following equation

$$\begin{cases} \dot{e}(t) = (A + LC)e(t) + 2LCx(t) \\ e(0) = z_0 - x_0 := e_0 \end{cases} \tag{4}$$

3. Main result

1. Consider the positive system given by (1). If $G = gI, \quad g \in \mathbb{R}_+^*$, where I is the identity operator of H, is an operator such that $g < \frac{\omega}{\beta M}$, then the system given by

$$\begin{cases} \dot{z}(t) = (A + GC^*C)z(t) + GC^*Cx(t) \\ z(0) = z_0 \end{cases} \tag{5}$$

is a positive observer of the system (1).

Where the positive constant β is such that : $\| C^*C \| \leq \beta$, M and ω are given by (2).

2. If the system given by (3) is a positive observer with gain $L = gC^*$, then for every positive linear operator L_1 that satisfying the condition $L_1 \leq L$, the following linear system

$$\begin{cases} \dot{z}(t) = (A + L_1C)z(t) + L_1Cx(t) \\ z(0) = z_0 \end{cases} \tag{6}$$

is also a positive observer of the system given by (1).

4. Application to an isothermal Plug-Flow (bio) reactor

The dynamics of the process in such reactor are described by the following PDEs:

$$\begin{cases} \frac{\partial x_1(s,t)}{\partial t} = -\nu \frac{\partial x_1(s,t)}{\partial s} - k_0 x_1(s,t) \\ \frac{\partial x_2(s,t)}{\partial t} = -\nu \frac{\partial x_2(s,t)}{\partial s} + bk_0 x_1(s,t) \\ x_1(s=0,t) = 0, \qquad x_2(s=0,t) = 0 \\ x_1(s,t=0) = x_{10}, \qquad x_2(s,t=0) = x_{20} \end{cases} \tag{7}$$

The model given by (7), can be reformulated as an abstract linear differential equation on the space $H = L^2[0,1] \times L^2[0,1]$ on the following form

$$\begin{cases} \dot{x}(t) = \tilde{A}x(t) \qquad , x(0) = x_0 \\ y(t) = Cx(t) \end{cases} \tag{8}$$

Where $x(t) = \begin{pmatrix} x_1(\cdot,t) \\ x_2(\cdot,t) \end{pmatrix}$ and $\tilde{A}$ is the linear operator defined by

$$\tilde{A} := \begin{pmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{pmatrix} := \begin{pmatrix} -\nu\frac{d\cdot}{ds} - k_0 I & 0 \\ bk_0 & -\nu\frac{d\cdot}{ds} \end{pmatrix} \tag{9}$$

And C is a linear operator defined by

$$Cx := \int_0^1 \mathbb{1}_{[1-\epsilon,1]} x(s) ds \tag{10}$$

The observer (11) has the following explicit form:

$$\begin{cases} \frac{\partial z_1(s,t)}{\partial t} = -\nu \frac{\partial z_1(s,t)}{\partial s} - k_0 z_1(s,t) + gC^*{}_1C_1z_1(s,t) + gC^*{}_1C_1x_1(s,t) \\ \frac{\partial z_2(s,t)}{\partial t} = -\nu \frac{\partial z_2(s,t)}{\partial s} + bk_0 z_1(s,t) + gC^*{}_2C_2z_2(s,t) + gC^*{}_2C_2x_2(s,t) \\ z_1(s=0,t) = 0, \qquad z_2(s=0,t) = 0 \\ z_1(s,t=0) = z_{10}, \qquad z_2(s,t=0) = z_{20} \end{cases} \tag{11}$$

5. Simulation results

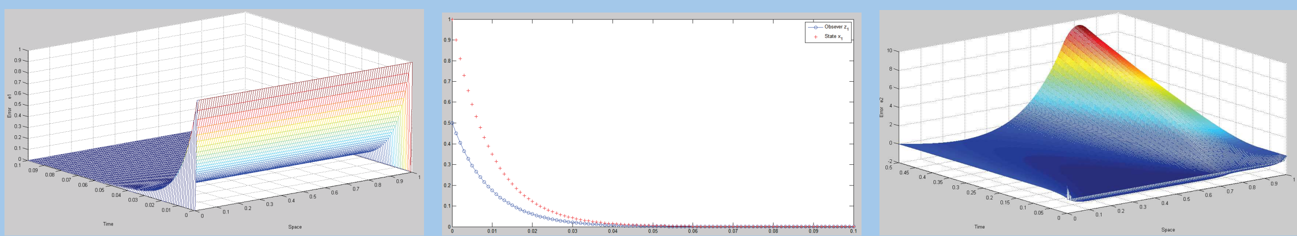


Figure 2: *Evolution of the error on reactant and product concentrations*

6. Discussions & conclusion

1. The choice of the observer gain $L = gC^*$ in such that g belong to the interval $\left]0, \frac{\omega}{M\beta}\right[$ is limited, as a consequence the choice of the observer gain is also limited.
2. If A is a generator of a positive contraction C_0-semi group exponentially stable, then we show that we can set $M = 1$ in the result 1.
3. If A is a generator of contraction C_0-semi group with an uniformly dissipative generator i.e $\langle Ax, x\rangle \leq -\gamma\|x\|^2$, $\forall x \in D(A)$, then we show that we can set $M = 1$ in the result 1.
4. The main result has been successfully applied to a chemical Plug-Flow reactor model.
5. The problem of extending the approach developed in this paper to a class of positive semilinear distributed parameter systems is currently under investigation.

7. References

[1] M. E. Achhab, On observers and compensators for infinite dimensional semilinear systems, Evolution Equations and Control Theory, Volume 4, Number 2, (2015)

[2] R. F. Curtain and H. J. Zwart, *An Introduction to Infinite-Dimensional Linear Systems Theory*, Springer-Verlag, (1995)

[3] R. Gressang, and G. Lamont, Observers for systems characterized by semi groups. IEEE Trans. Autom.Control, AC-20, 523–528 (1975).

Output reachability and output controllability of positive linear discrete delay systems

M. NAIM*, F. LAHMIDI*, A. NAMIR** and M. RACHIK*
naimmouhcine2013@gmail.com
*Laboratory Analyse, Modelization and Simulation, University Hassan II of Casablanca
**Laboratory of Information Technology and Modeling, University Hassan II of Casablanca

Introduction

Positive systems are a wide class of systems in which state variables and outputs are constrained to be positive, or at least nonnegative for all time whenever the initial states and inputs are nonnegative. Many practical systems in engineering, management science, economics, social sciences, compartmental analysis in biology and medicine, can be meddled as positive systems. In this work, necessary and sufficient conditions for output reachability and null output controllability of positive linear discrete systems with delays in state, input and output are established, and we show that output reachability and null output controllability together imply output controllability.

Problematic

We consider the positive discrete linear delay system

$$\begin{cases} x_{i+1} = \sum_{j=0}^{p} A_j x_{i-j} + \sum_{j=0}^{q} B_j u_{i-j}\,, i \in \mathbb{N}, \\ u_{-j} \in \mathbb{R}_+^m \; for\; j \in \sigma_1^q \; and \; x_{-j} \in \mathbb{R}_+^n \; for \; j \in \sigma_0^p, \end{cases} \qquad (1)$$

with the output equation

$$y_i = \sum_{j=0}^{l} C_j x_{i-j} + \sum_{j=0}^{v} D_j u_{i-j}\,, \qquad (2)$$

where $x_i \in \mathbb{R}_+^n$ is the system state, $u_i \in \mathbb{R}_+^m$ is the input, $y_i \in \mathbb{R}_+^r$, $A_j \in \mathbb{R}_+^{n\times n}$ $(j \in \sigma_0^p)$ are the matrices of the state, $B_j \in \mathbb{R}_+^{n\times m}$ $(j \in \sigma_0^q)$ are the matrices of the input, $C_j \in \mathbb{R}_+^{r\times n}$ $(j \in \sigma_0^l)$ are the matrices of the output, $D_j \in \mathbb{R}_+^{r\times m}$ $(j \in \sigma_0^v)$ are the matrices of the feedthrough, $\sigma_k^s = \{k, k+1, \dots, s\}$, with $k \le s$, $k, s \in \mathbb{N}$.

In this work we discuss a fundamental question for dynamic, positive systems modeled by (1) and (2). Is it possible to transfer the output of the system from a given initial value to any other output?

We prove that for any $i \in \mathbb{N}$, we have

$$y_i = \mathcal{Q}_{i+1}\tilde{x} + \mathcal{R}_i u_0^{i+1},$$

with

$$\mathcal{Q}_i \in \mathbb{R}_+^{r\times(n(p+1)+mq)},$$

$$\tilde{x} = \left(x_0, x_{-1}, \dots, x_{-p}, u_{-1}, \dots, u_{-q}\right)^T \in \mathbb{R}_+^{n(p+1)+mq},$$

$$\mathcal{R}_i \in \mathbb{R}_+^{r\times(i+1)m},$$

and

$$u_0^{i+1} = (u_0, u_1, \dots, u_i)^T \in \mathbb{R}_+^{(i+1)m}.$$

Results and discussions

Output reachability

Definition. The positive system (1), (2) is said to be output reachable if there exists $N \in \mathbb{N}_+$ such that for any final output $y_f \in \mathbb{R}_+^r$, there exists an input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$, which steers the output of the system fom $x_{-j} = 0, j \in \sigma_0^p$, with $u_{-j} = 0$ for $j \in \sigma_1^q$ to y_f.

Theorem. The positive system (1), (2) is output reachable if and only if for some $N \in \mathbb{N}_+$, the output reachability matrix $\mathcal{R}_N$ includes a monomial submatrix of order $r \times r$.

Null output controllability

Definition. The positive system (1), (2) is said to be null output controllable if there exists $N \in \mathbb{N}_+$ such that for any initial state $x_{-j} \in \mathbb{R}_+^n, j \in \sigma_0^p$ and any initial input $u_{-j} \in \mathbb{R}_+^m, j \in \sigma_1^q$, there exists an input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$, which steers the output of the system fom x_{-j} to zero.

Theorem. The positive system (1), (2) is null output controllable if and only if for some $N \in \mathbb{N}_+$, the null output controllablity matrix $\mathcal{Q}_N$ is null.

Output controllability

Definition. The positive system (1), (2) is said to be output controllable if there exists $N \in \mathbb{N}_+$ such that for any nonnegative initial state $x_{-j} \in \mathbb{R}_+^n, j \in \sigma_0^p$ and any initial input $u_{-j} \in \mathbb{R}_+^m, j \in \sigma_1^q$, there exists an input sequence $u_i \in \mathbb{R}_+^m$, $i \in \sigma_0^{N-1}$ which steers the output of the system fom x_{-j} to the desired final output $y_f \in \mathbb{R}_+^r$.

Theorem. The positive system (1), (2) is output controllable if and only if it is output reachable and null output controllable.

In many engineering applications, it is needed to direct the output of the systems toward some desired value

The output controllability of positive discrete linear systems with delays has been considered. Necessary and sufficient conditions for the positivity of discrete systems have been established. Criteria for output reachability and null output controllability of the positive discrete systems have been also proved. It has been shown that output reachability and null output controllability together imply output controllability.

جامعة شعيب الدكالي
Université Chouaïb Doukkali

Positive State Reachability of Discrete time Positive Linear Systems

[1]OUYADRI Mourad, [1]LAABISSI Mohamed, [1]ACHHAB Elarbi
[1]Laboratoire d'Informatiques et Mathématiques et Leurs Applications (LIMA), Faculté des sciences d'El Jadida, Maroc

Faculté des Sciences El Jadida

Abstract

Controllability of positive system s is applied only when the state and input are nonnegative for all time. So in the applications, the need of positive input is often overly restrictive. In this paper we study a new vision of positive state controllability of positive systems. We use a connection between positive state controllability and positive input controllability of a related system, wish help us to obtain a controllability criteria like Kalman. Finally, we apply this study in an example.

Key words : Controllability, Reachability, Linear System, Discrete Time, Positive System, Positive State Reachability, Positive State Controllable, Positive Input Controllability.

Introduction

The concept of controllability introduces the ability of dynamic systems to have its state driven from any initial state to any final state , so controllability of dynamic systems prove that if a controller can be applied to generate a desired state space behaviour. For linear systems the formulation of controllability concept dates back to Kalman [2]. But the problem is that, controllability does not respect any nonnegativity of systems, where the state and input variables correspond to nonnegative quantities. This problem motivited the development of positive systems, were now existe several textbooks on this subject (for exampl [3,4,5]).

Controllability of positive systems, that is positive input controllability is more limited than the general case, but the situation is well understood (see [6,7]) that the important in the positive systems is that the state and input variables must both be nonnegative.

Under the constraint that just the state must be nonnegative, we call that positive state controllability [1], there are many papers (for example [8,9,10,11]) were the state is nonnegative but the control u can take a negative value. The study of positive state controllability is in the positive cone of the space, and not in the space of the input and state, so it is not clear that the positive input controllability is applicable. So the important key that this paper is based on,is that the positive state controllability is equivalent to positive input controllability of a related positive system, under certain assumption. By using this approach, we characterise the Reachability of systems were the input can take a negative value, under the constuction that the state must remain nonnegative.

Problem and result

Let the controled system

$$x(k+1) = Ax(k) + Bu(k)$$

Where x , A , b are nonnegative , the control u can take a negative value. So we can't apply the controllability theory of positive systems (positive input controllability) to this system because the controler can take a negative value. The solution of this problem is treated by Guiver & all [1], they used the following assumption :

$$(H)\ \begin{cases} \textbf{\textit{Given the pair }} (A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \textbf{\textit{ with }} A, B \geq 0 \textbf{\textit{ there exists}} \\ F \in \mathbb{R}^{m\times n} \textbf{\textit{ such that with }} \tilde{A} = A - BF \textbf{\textit{ both }} \tilde{A} \geq 0 \textbf{\textit{ and if }} v \in \mathbb{R}^{n}{}_{+} \\ w \in \mathbb{R}^{m} \textbf{\textit{ satisfy }} \tilde{A}v + Bw \geq 0 \textbf{\textit{ then }} w \geq 0, \end{cases}$$

To prove that : The state trajectories of (A, B) from the initial state $x_0 \in \mathbb{R}^n{}_+$ with nonnegative state are precisely the state trajectories of $(\tilde{A}, B)$ from the initial state $x_0 \in \mathbb{R}^n{}_+$ <u>with nonnegative control</u>, and conserving the nonnegativity of state.

The consequence of this, is that (under assumption (H)) results for positive state controllability of the pair (A, B) follow from existing results for positive input controllability for the pair $(\tilde{A}, B)$.

In this case we can apply the controllability theory of positive system (positive input controllability) to the related positive system $(\tilde{A}, B)$ and that is equivalent to positive state controlabillity of (A, B) under the assumption (H), so by using Kalman's controllability theory, we conclude the following corollary :

Corollary:

Under the assumption (H),
The pair (A,B) is positive state contollability if and only if the matrix $[B\ \tilde{A}B\ \dots\ \tilde{A}^{n-1}B]$ is monomial.

But, all this is based only on the existence of matrix $\boldsymbol{F}$ (with contraints) wish gives a relative positive system with positive matrix $\tilde{A} = A - BF$ and positive control $\tilde{u}$, that give us the ability to check the positive input controllability. In this paper we try to find a new characterisation of positive state controllability of (A, B), were we don't have the need of a related positive system, So we started by the characterisation of positive state reachability of system (A, b) with simple input u, as result we obtain the following theorem:

Theorem:

The pair (A, b) is positive state reachability if and only if the reachability matrix
$[b\ Ab\ \dots\ A^{n-1}b] = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \alpha_{n,n} \end{bmatrix}$ **is upper triangular such that** $\alpha_{i,i} > 0\ \forall\ i = 1 \dots n$,
$and\ \alpha_{1,1} \times \alpha_{j,k+1} = \alpha_{1,k-j+1} \times \alpha_{j,j}\ \forall j \leq k\ , j = 1 \dots n$.

Finally we apply this result on the following example:

Exemple:

Let the pair (A, b) given by $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, and $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

The reachability matrix $[b\ Ab\ A^2b] = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$ of (A, b) is upper triangular, and $\alpha_{i,i} > 0\ \forall\ i = 1 \dots n$
$and\ \alpha_{1,1} \times \alpha_{j,k+1} = \alpha_{1,k-j+1} \times \alpha_{j,j}\ \forall j \leq k\ , j = 1 \dots n$ are verified.
So -after the above theorem- the pair (A, b) $is\ positive\ state\ reachability$.
We can check that, by using the results in [1].

We obtain $f = [1\ 1\ 1]$ wish gives $\tilde{A} = A - bf = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \geq 0$ such that the relative pair $(\tilde{A}, b)$ is a positive system, and the reachability matrix $[b\ \tilde{A}b\ \tilde{A}^2b] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is monomial ,
so we conclude -after the above corollary- that the pair (A, b) $is\ positive\ state\ reachability$.

References

[1] C.Guiver, D.Hodgson, S.Townley, Positive state controllability of positive linear systems, Systems & Control Letters 65 (2014) 23–29,
[2] R. Kalman, On the general theory of control systems, IRE Trans. Automat.Control 4 (3) (1959) 481–492.
[3] A. Berman, M. Neumann, R.J. Stern, Nonnegative Matrices in Dynamic Systems,in: Pure and Applied Mathematics (New York), John Wiley & Sons Inc., NewYork, 1989.
[4] M.A. Krasnoselśkij, J.A. Lifshits, A.V. Sobolev, Positive Linear Systems, in: Sigma Series in Applied Mathematics, vol. 5, Heldermann Verlag, Berlin, 1989.
[5] L. Farina, S. Rinaldi, Positive Linear Systems, in: Pure and Applied Mathematics (New York), Wiley-Interscience, New York, 2000.
[6] P.G. Coxson, H. Shapiro, Positive input reachability and controllability of positive systems, Linear Algebra Appl. 94 (1987) 35–53.
[7] L. Caccetta, V.G. Rumchev, A survey of reachability and controllability for positive linear systems, Ann. Oper. Res. 98 (2000) 101–122.
[8] D. Doak, P. Kareiva, B. Klepetka, Modeling population viability for the desert tortoise in the western mojave desert, Ecol. Appl. 4 (3) (1994) 446–460.
[9] K.R. Crooks, M. Sanjayan, D.F. Doak, New insights on cheetah conservation through demographic modeling, Conserv. Biol. 12 (4) (1998) 889–895.
[10] P. Kareiva, M. Marvier, M. McClure, Recovery and management options for spring/summer chinook salmon in the columbia river basin, Science 290 (5493) (2000) 977–979.
[11] J. Lubben, B. Tenhumberg, A. Tyre, R. Rebarber, Management recommendations based on matrix projection models: the importance of considering biological limits, Biol. Conserv. 141 (2) (2008) 517–523.